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Structural Mechanics

Forced Vibration of Out of Plane Loaded Stepped Circular Rods

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ABSTRACT

In this paper, we investigate the forced vibration of out of plane loaded stepped circular rods subjected to various time depended loads in the Laplace domain. The material of the rods is assumed to be homogeneous, isotropic and elastic. The effect of shear deformation is also taken into account. Laplace transform is applied to the obtained governing equations in the time domain. Subsequently, the Complementary Functions Method (CFM) is used to solve the attained canonical form of the first order ordinary differential equations in the transformed domain. Thereafter the related differential equations are solved by the fifth-order Runge – Kutta (RK5) method. To retransfer the results to the time domain an efficient inverse numerical Laplace transform method is implemented. To validate and compare the results of the present method a computer program is coded in Fortran. Verification and exactness of the written program is performed by comparing the results of the present methods and results of ANSYS which is a commercial finite element program, the accuracy and superiority of the present method can be noticed.

Keywords: Two-Point Boundary Value Problems; Complementary Functions Method (CFM); Inverse Laplace Transforms; Stepped Circular Rods.

1. INTRODUCTION

Out of plane curved rods are commonly used as structural elements in various branches of engineering such as automotive, mechanical and civil engineering. Since the practical applications of curved rods as above engineering structures, their static and dynamic behaviour is investigated by many researchers.
Haktanır [1] investigated the static behavior of in-plane rods, made up of isotropic-elastic material, by stiffness matrix method based on Complementary Functions Method (CFM). The Runge–Kutta fourth-order method has been used for the solution of the obtained equations. Bayhan [2] studied the static behavior of planar frames with members of circular axes with the aid of both transfer and stiffness matrix methods. Bozkurt [3] used Complementary Functions Method (CFM) for the bending analysis of circular planar systems, helicoidal stair cases, axisymmetric shell structures, and cylindrical vault structures under static loadings. Yildirim, İnce and Kiral [4] investigated the static analysis of compound planar frames with members of linear and circular axes by stiffness matrix method. The stiffness matrixes and force vectors of the frame members with circular axes, under planar loads and perpendicular loads to plane, were calculated by transform matrix method precisely.

The dynamic behaviour of cylindrical helical rods made of isotropic, anisotropic and elastic, viscoelastic materials under time dependent loads is investigated by Çalım [5] in the Laplace domain theoretically. In the solutions, Kelvin model was employed. The obtained solutions transformed to the time domain by using a proper inverse Laplace transform method. Kıraç [6] examined the dynamic behaviour of composite straight rods subjected to time-dependent loads in the Laplace domain theoretically. The free vibration was considered as a special case of forced vibration. In the formulations, the effect of the rotary inertia, axial and shear deformations were taken into account. The ordinary differential equations obtained in canonical form in the Laplace domain were solved numerically by the Complementary Functions Method to calculate the dynamic stiffness matrix. Aktan [7] studied the free vibration of in-plane circular beams. The Timoshenko beam theory was used and the effect of the rotary inertia was considered. Çoban [8] investigated the dynamic analysis of curved beams using mixed finite element method with Gâteaux derivative. Akkurt [9] investigated the dynamic behaviour of straight and circular rods resting on elastic foundation in the Laplace domain theoretically. Karaca [10] studied the static and dynamic analysis of circular Timoshenko rods which are loaded in-plane and out-of-plane, theoretically. The longitudinal vibrations of elastic bars were investigated using the Gâteaux differential method and the mixed finite element method by Ecer [11].

Free out-of-plane vibrations of a circular arch with uniform cross-section are investigated by Tufekci and Dogruer [12] while the effects of transverse shear and rotatory inertia are taking into account due to both flexural and torsional vibrations. Viola, Dilena and Tornabene [13] investigated the in-plane linear dynamic behaviour of multi-stepped and multi-damaged circular arches under different boundary conditions. Dönmez [14] gave an approximate solution to the governing equations of the out-of-plane and in-plane dynamic problems of a curved beam with varying cross-sections. An exact solution of free in-plane vibrations of circular arches of uniform cross-section is given by Tufekci and Arpaci [15]. A finite element approximation for static and free vibration problems of circular arches with continuously varying curvature and cross-section is presented by Eroğlu [16]. Tüfekci and Özdemirci [17] investigated the free in-plane vibration of stepped circular arches. The effects of axial extension, transverse shear deformation and rotatory inertia are included in the governing equations. The solution was obtained exactly by using the initial value method.

Literature survey shows that many studies dealing with static and free vibration analysis of curved rods have been made and several methods have been investigated by researchers. However, there are limited studies in the literature addressing to the solutions of out of plane curved rods subjected to dynamic loads by the CFM in the Laplace domain. In present paper the forced vibration of out of plane stepped curved rods has been analysed under various types of dynamic loads by Laplace transform method and the CFM. To the knowledge of the authors this unified method is used to determine the dynamic behaviour of the out of plane loaded curved rods for the first time. The CFM is a numerical solution method which transforms a two point boundary value problem to a system of initial value problems. For the solution of initial value problems RK5 is applied in this study. The application of Laplace Transform, with respect to time, to partial differential equations, converts them to ordinary differential equations in the transformed domain. Thus the numerical solution of partial differential equations in the Laplace domain can be done easily. To transform the obtained solutions in the Laplace domain to the time domain an efficient inverse Laplace transform method has been used. The results of the present method are compared with those solutions obtained from ANSYS and it has been seen that the results obtained in this study are found to be more exact and accurate.

2. MATERIAL AND METHODS

2.1 Governing equations

The partial differential equations of out of plane loaded stepped circular rods under dynamic loads are given as follows:

$$\frac{\partial U_b}{\partial \phi} = -r\Omega_n + r\frac{T_b \alpha_b}{GA(\phi)} \quad (1) \qquad \frac{\partial \Omega_t}{\partial \phi} = \Omega_n + r\frac{M_t}{GI_t(\phi)} \quad (2)$$

$$\frac{\partial \Omega_n}{\partial \phi} = -\Omega_t + r\frac{M_n}{EI_n(\phi)} \quad (3) \qquad \frac{\partial T_b}{\partial \phi} = r\rho A(\phi)\frac{\partial^2 U_b}{\partial t^2} - rp_b \quad (4)$$

$$\frac{\partial M_t}{\partial \phi} = r\rho I_t(\phi)\frac{\partial^2 \Omega_t}{\partial t^2} + M_n - rm_t \quad (5) \qquad \frac{\partial M_n}{\partial \phi} = r\rho I_n(\phi)\frac{\partial^2 \Omega_n}{\partial t^2} - M_t + rT_b - rm_n \quad (6)$$

$$A(\phi) = \pi h(\phi)^2 \quad (7) \qquad I_t(\phi) = \frac{\pi h(\phi)^4}{2}, I_n(\phi) = \frac{\pi h(\phi)^4}{4} \quad (8)$$

Here, E, ρ, $h(\phi)$,$A(\phi)$, $I_t(\phi)$, $I_n(\phi)$, α_n and r indicate the modulus of elasticity, mass density, radius of the cross-section, cross-section area, torsional moment of inertia, bending moment of inertia, shear correction factor and radius of curvature of the rod respectively.
The unknown column matrix, $\{Y(\phi,t)\}$, for the forced vibration of out of plane loaded stepped circular rods is given as:

$$\{Y(\phi,t)\} = \{U_b, \Omega_t, \Omega_n, T_b, M_t, M_n\}^T \quad (9)$$

The Laplace transform of a time-dependent function *f(t)* is the function "$\overline{F}(s)$".

$$L[f(t)] = \overline{F}(s) = \int_0^\infty f(t)e^{-st}dt \quad (10)$$

where s is the Laplace variable also known as operator variable in the Laplace domain. The Laplace transform of first and second derivatives with respect to time are as follows:

$$L\left[\dot{f}(t)\right] = s\overline{F}(s) - f(0) \quad (11)$$

$$L\left[\ddot{f}(t)\right] = s^2\overline{F}(s) - sf(0) - \dot{f}(0) \quad (12)$$

Applying the Laplace transform to equations (1-6), converts these partial differential equations to variable-coefficient ordinary differential equations. Thereby, the governing ordinary differential equations of the dynamic behavior of in-plane loaded stepped circular arches can be obtained in the Laplace domain as follows:

$$\frac{d\overline{U}_b}{d\phi} = -r\overline{\Omega}_n + r\frac{\overline{T}_b \alpha_b}{GA(\phi)} \quad (13)$$

$$\frac{d\overline{\Omega}_t}{d\phi} = \overline{\Omega}_n + r\frac{\overline{M}_t}{GI_t(\phi)} \quad (14)$$

$$\frac{d\overline{\Omega}_n}{d\phi} = -\overline{\Omega}_t + r\frac{\overline{M}_n}{EI_n(\phi)} \tag{15}$$

$$\frac{d\overline{T}_b}{d\phi} = rs^2\rho A(\phi)\overline{U}_b - r\overline{p}_b \tag{16}$$

$$\frac{d\overline{M}_t}{d\phi} = rs^2\rho I_t(\phi)\overline{\Omega}_t + \overline{M}_n - r\overline{m}_t \tag{17}$$

$$\frac{d\overline{M}_n}{d\phi} = rs^2\rho I_n(\phi)\overline{\Omega}_n - \overline{M}_t + r\overline{T}_b - r\overline{m}_n \tag{18}$$

Where the terms shown by $(\overline{\bullet})$ indicates the Laplace transform of the quantities.

$$\begin{aligned}
L\left[\rho A(\phi)\frac{\partial^2 U_b}{\partial t^2}\right] &= \rho A(\phi)\left[s^2\overline{U}_b - sU_b(\phi,0) - \frac{\partial U_b(\phi,0)}{\partial t}\right] \\
L\left[\rho I_t(\phi)\frac{\partial^2 \Omega_t}{\partial t^2}\right] &= \rho I_t(\phi)\left[s^2\overline{\Omega}_t - s\Omega_t(\phi,0) - \frac{\partial \Omega_t(\phi,0)}{\partial t}\right] \\
L\left[\rho I_n(\phi)\frac{\partial^2 \Omega_n}{\partial t^2}\right] &= \rho I_n(\phi)\left[s^2\overline{\Omega}_n - s\Omega_n(\phi,0) - \frac{\partial \Omega_n(\phi,0)}{\partial t}\right]
\end{aligned} \tag{19}$$

The second and third terms on the right-hand side of the equation (19) are the initial conditions given for t=0; in present study those terms are assumed to be zero.

2.2 Complementary functions method

The matrix notation of the ordinary differential equations (13-18) obtained in the Laplace domain is given below:

$$\frac{d\left\{\overline{Y}(\phi,s)\right\}}{d\phi} = \left[\overline{A}(\phi,s)\right]\left\{\overline{Y}(\phi,s)\right\} + \left\{\overline{F}(\phi,s)\right\} \tag{20}$$

Here ϕ is independent variable and s is the Laplace transform parameter. The state vector for in-plane stepped circular arches is given by equation (21).

$$\left\{\overline{Y}(\phi,s)\right\} = \left\{\overline{U}_b(\phi,s), \overline{\Omega}_t(\phi,s), \overline{\Omega}_n(\phi,s), \overline{T}_b(\phi,s), \overline{M}_t(\phi,s), \overline{M}_n(\phi,s)\right\}^T \tag{21}$$

The CFM is based on the principle of the solution of equation (20) with the help of the initial conditions. This method is basically the reduction of two-point boundary value problems to initial-value problems. The general solution of Eq. (20), is given by

$$\left\{Y(\phi,s)\right\} = \sum_{m=1}^{n} C_m\left[\overline{U}^{(m)}(\phi,s)\right] + \left\{\overline{V}(\phi,s)\right\} \tag{22}$$

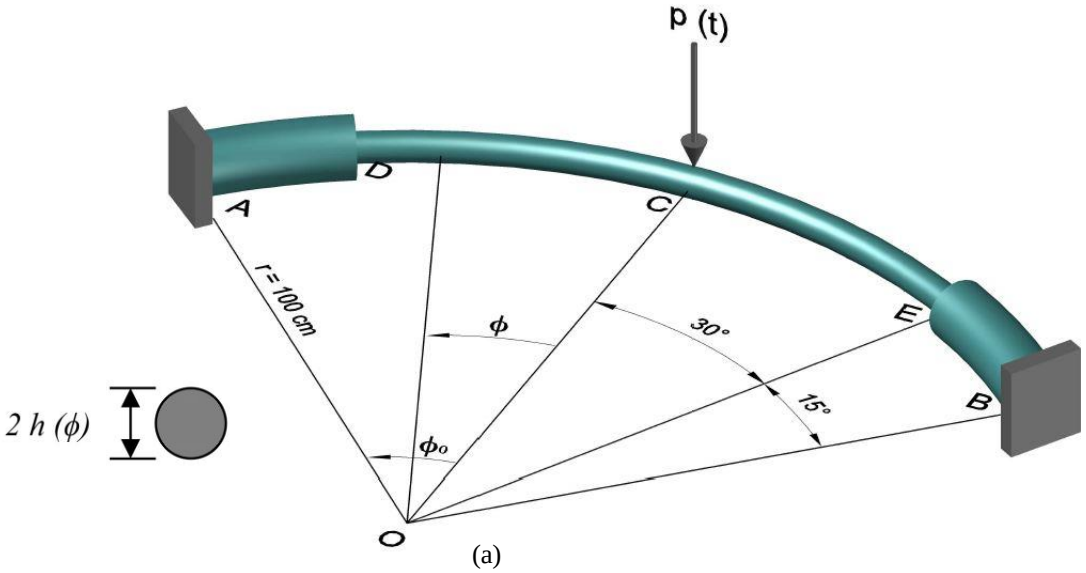
Where $\left[\overline{U}^{(m)}(\phi,s)\right]$ is the complementary solution such that its mth component is equal to 1, whereas all the others are zero. $\left\{\overline{V}(\phi,s)\right\}$ is the inhomogeneous solution with all zero initial conditions, the integration constants C_m will be determined from the boundary conditions at both ends.

The results, obtained in the Laplace domain, are transformed to the time domain with the help of modified Durbin's numerical inverse Laplace transform method given by Durbin [18] and Temel [19].

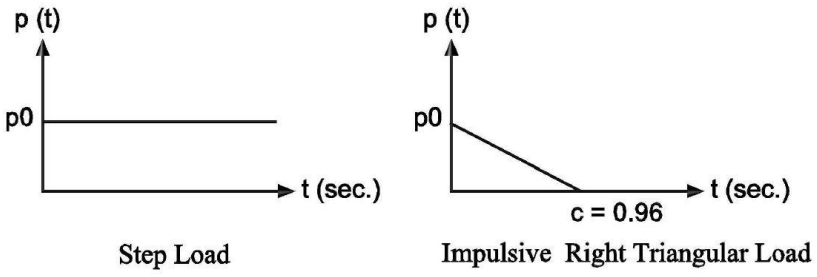
3. RESULTS AND DISCUSSIONS

A fixed-ended isotropic stepped circular rod, shown in figure 1.a, is now considered under an out of plane point load applied to its midpoint. Material properties density, ρ=7850x10^{-6} kg/cm^3, Poisson's ratio, ν = 0.3, and modulus of elasticity, E=2.1x10^6 kgf/cm^2. Two types of time depended point loads, shown in figure 1.b, with the amplitude $p0$ = 1 kgf are implemented to the arch. The equations (13-18) given in canonical form are solved numerically in the Laplace domain by the CFM. The results are compared with those obtained from ANSYS. Comparisons are shown in the graphics. In this problem the effect of shear deformation is taken into account. Here the torsional, flexural rigidity and cross section of the rod are;

$$GI_t(\phi) = G\frac{\pi h(\phi)^4}{2};\ EI_n(\phi) = E\frac{\pi h(\phi)^4}{4};\ A(\phi) = \pi h(\phi)^2$$



(a)



(b)

Figure 1. (a) Fixed –ended circular stepped rod; (b) Dynamic loads.

The radius of the cross section ($h(\phi)$) and the boundary conditions of fixed-end and symmetric point of the rod are considered to be;

$$h(\phi)=\begin{cases}0.5 & , \quad -\dfrac{\pi}{6}<\phi<\dfrac{\pi}{6}\\ 1.0 & , \quad \dfrac{\pi}{6}\le\phi\le\dfrac{\pi}{4}\\ 1.0 & , \quad -\dfrac{\pi}{4}\le\phi\le-\dfrac{\pi}{6}\end{cases} \qquad ; \qquad \phi=0\rightarrow\begin{cases}\Omega_n=0\\ T_b=p/2\\ M_t=0\end{cases} \qquad ; \qquad \phi=\phi_0\rightarrow\begin{cases}U_b=0\\ \Omega_t=0\\ \Omega_n=0\end{cases}$$

The geometric properties of the rod are as follows:

Shear correction factor $\alpha_b = 1.11$, radius of the circular rod $r = 100$ cm and $\phi_0 = \pi/4$.

$$h=0.5cm\rightarrow\begin{cases}A=0.7854cm^2\\ I_t=0.0982\text{cm}^4\\ I_n=0.0491\text{cm}^4\end{cases} \qquad ; \qquad h=1.0cm\rightarrow\begin{cases}A=3.1416cm^2\\ I_t=1.5708\text{cm}^4\\ I_n=0.7854\text{cm}^4\end{cases}$$

Laplace transforms of the step load and the impulsive right triangular load are available in closed-form. The vertical U_b displacement of the midpoint of the rod under a point step load obtained by present method is presented in Figs. 2.

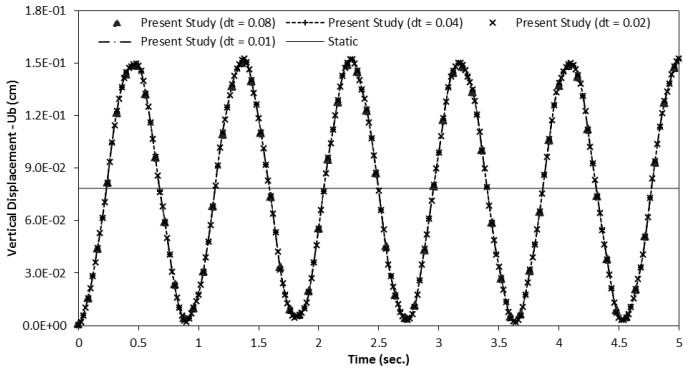


Figure 2. U_b Vertical displacement of the midpoint of the rod versus time under a point step load.

The vertical shear force T_b results obtained by the present study at the fixed-end of the rod subjected to a point step load is presented in Fig. 3.

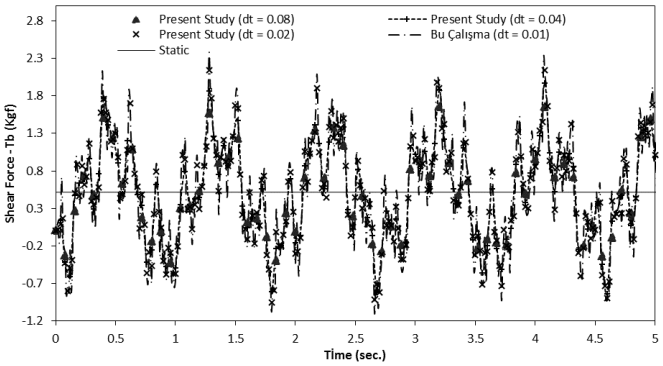


Figure 3. T_b Vertical shear force at the fixed end of the rod versus time under a point step load.

A uniaxial element is used to analyze the circular rod problem in ANSYS. Time varying results of the U_b vertical displacement of the symmetric point and T_b shear force under point step load at the fixed support are given in Figs. 4 – 5.

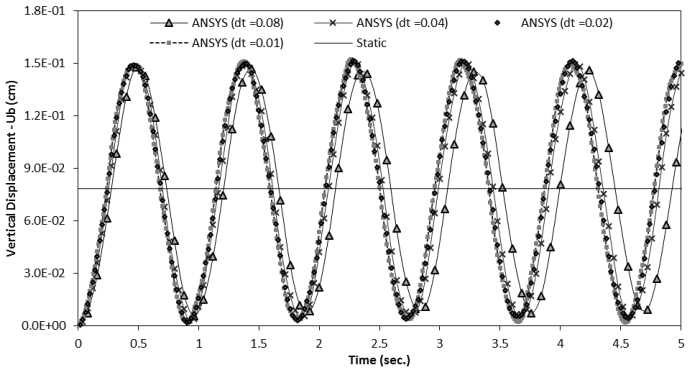


Figure 4. U_b Vertical displacement of the midpoint of the rod versus time under a point step load.

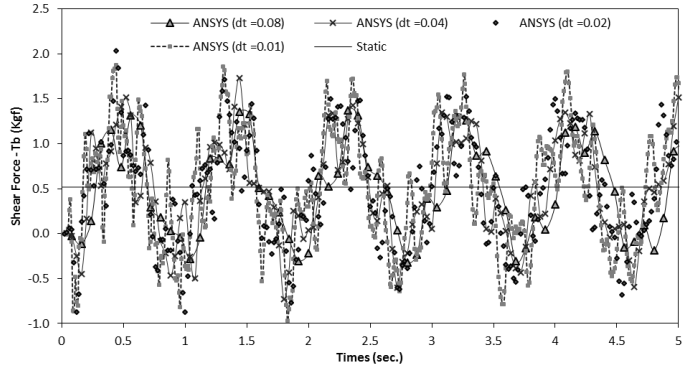


Figure 5. T_b Vertical shear force at the fixed end of the rod versus time under a point step load.

The problem has been solved for various time increments as time steps (dt=0.08 sec., N = 64), (dt =0.04 sec., N = 128), (dt =0.02 sec., N = 256) and (dt = 0.01 sec., N = 512). It is apparent that results obtained for a coarse time increment along with fewer Laplace transform parameters overlap the results obtained with finer increments and higher parameters. This indicates the efficiency and exactness of the present method.

As it is obvious in the figures above that the accuracy of the results of conventional numerical time integration methods depends on the appropriate selection of the optimum time increment. Thus a small time increment is needed to obtain efficient results.

The comparison of present study and conventional step by step time integration methods for various time increments and stepped cross section given in Figs. 6 – 8 show the U_b vertical displacement, M_n bending moment at the midpoint and T_b shear force at the fixed supported end of the rod for a step load respectively.

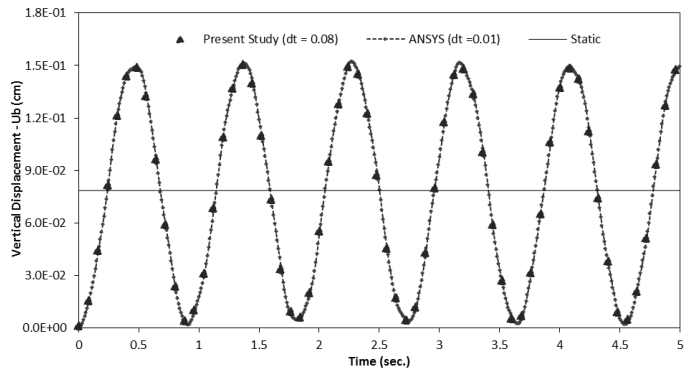


Figure 6. Comparison of the U_b Vertical displacement of the midpoint of the rod under a point step load.

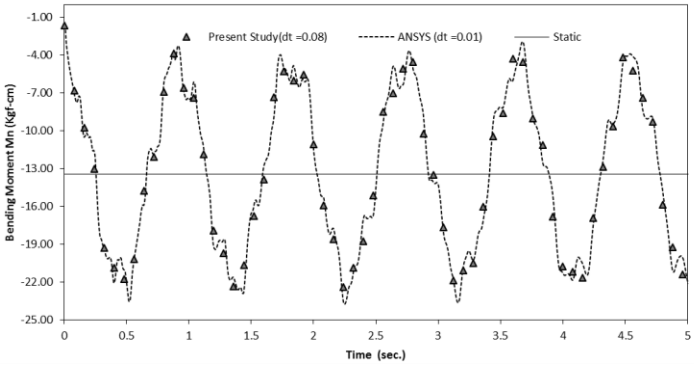


Figure 7. Comparison of the M_n Bending moment of the midpoint of the rod under a point step load.

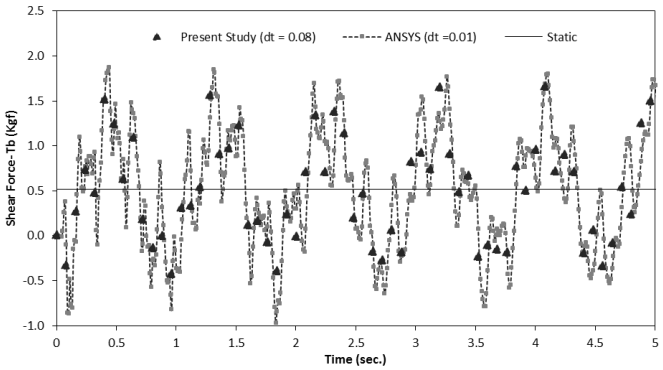


Figure 8. Comparison of T_b Vertical shear force at the fixed end of the rod versus time under a point step load.

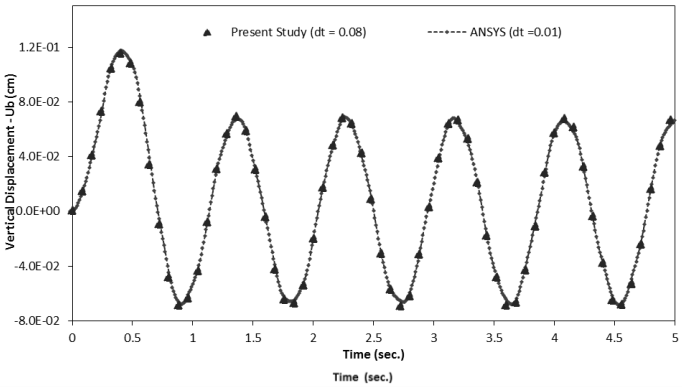


Figure 9. Comparison of the U_b Vertical displacement of the midpoint of the rod under a point impulsive right triangular dynamic load.

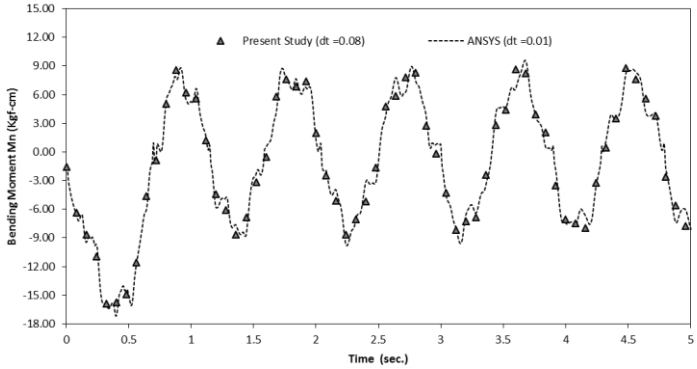


Figure 10. Comparison of the M_n Bending moment of the midpoint of the rod under a point impulsive right dynamic load.

To examine the dynamic response of this structure under different time depended loads the impulsive right triangular dynamic point load is applied to the stepped circular rod. Solutions are obtained in the similar manner as step load and comparisons are given in Figs. 9 - 12.

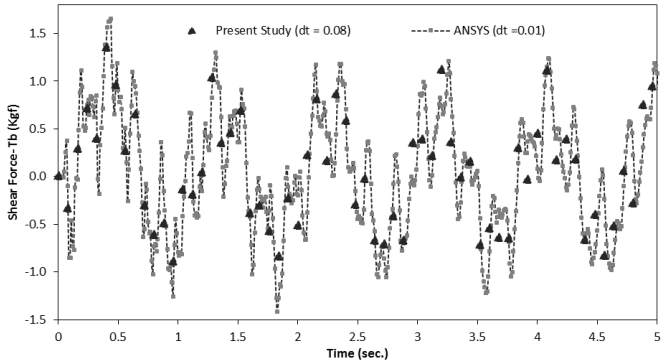


Figure 11. Comparison of T_b Vertical shear force at the fixed end of the rod versus time under a point impulsive right triangular dynamic load.

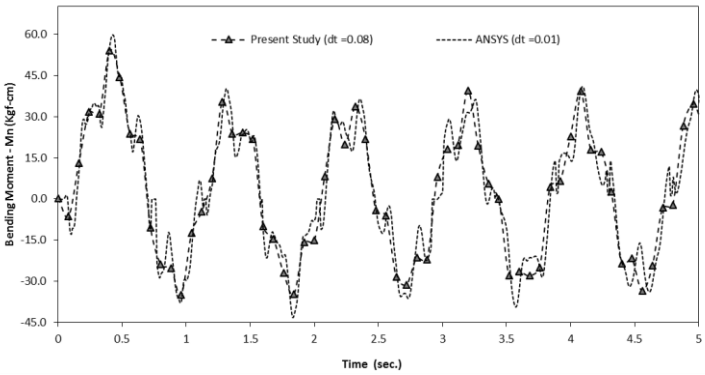


Figure 12. Comparison of M_n Bending moment at the fixed end of the rod versus time under a point impulsive right triangular dynamic load.

4.CONCLUSION

This paper investigates the transient analysis of linear elastic stepped circular rods under various types of time depended loads. The dynamic behaviour of such structures is examined by the unified method of Laplace transform and the CFM as the effect of shear deformation is considered. RK5 algorithm is used for the numerical solution of the initial value problems. The governing equations of the related problem are first obtained in the time domain. Laplace transform is then applied and the set of simultaneous linear algebraic equations are solved by the CFM in the Laplace domain for a set of Laplace parameters. The solutions obtained are transformed to the time domain using an efficient inverse numerical Laplace transform method. For the suggested model, a computer program is coded in Fortran. Results of the presented method are compared with those obtained from ANSYS.

The accuracy of the results of conventional step by step time integration method depends on the appropriate selection of the optimum time steps. By using the presented method, highly accurate results can be obtained, even with a coarse time steps. It is manifest that combination of Laplace transform and the CFM is far more efficient than the conventional step-by-step time integration methods. Laplace transformation gives a time-independent boundary-value problem in spatial coordinate which is then solved by the CFM. The numerical examples has proved that the suggested procedure is highly accurate and efficient compared to various other numerical methods available in the literature and it can be easily applied to the stepped circular rods.

REFERENCES

[1] V. Haktanir, 1990. *Investigation of the static, dynamic and buckling behavior of cylindrical helical rods by transfer matrix method*, PhD. Thesis, University of Cukurova, Adana.

[2] S. Bayhan, 1993. *Analysis of planner curved rods by stiffness matrix and transfer matrix method*, MSc. Thesis, University of Cukurova, Adana (In Turkish).

[3] M. Bozkurt, 1995. *The Analysis of cylindrical vaults, circular and helicoidal structural systems by the complementary functions method-Mathematica applications*, MSc. Thesis, University of Cukurova, Adana (In Turkish).

[4] V. Yildirim, N. İnce and , E. Kiral, 1997. *Static analysis of compound planar frames with members of linear and circular axes by stiffness matrix method*, J. of Engineering and Environmental Sciences, 137-148.

[5] F.F. Çalim, 2003. *Dynamic analysis of viscoelastic, anisotropic curved spatial rod systems*, PhD. Thesis, University of Cukurova, Adana(In Turkish).

[6] M. Kiraç, 2007. *Dynamic analysis of straight composite rods*, MSc. Thesis, University of Mustafa Kemal, Hatay (In Turkish).

[7] H. Aktan, 2008. *In-Plane dynamic analysis of circular beams*, MSc. Thesis, Istanbul Technical University, Istanbul (In Turkish).

[8] M. Çoban, 2008. *Out-Of-plane dynamic analysis of curved beams using mixed finite element method*, MSc. Thesis, Istanbul Technical University, Istanbul, (In Turkish).

[9] F.G. Akkurt, 2011. *Dynamic analysis of straight and circular rods on elastic foundation*, MSc. Thesis, University of Mustafa Kemal, Hatay (In Turkish).

[10] N. Karaca, 2014. *Static and dynamic analysis of planar rods by the transfer and stiffness matrix methods*," MSc. Thesis, University of Mustafa Kemal, Hatay (In Turkish).

[11] S. Ecer, 2015. *Longitudinal vibration analysis of straight bars using mixed finite element method*, MSc. Thesis, Istanbul Technical University, Istanbul, (In Turkish).

[12] E. Tüfekci and O. Doğruer, 2006. *Out of plane free vibration of a circular arch with uniform cross-section: Exact solution*, J. Sound and Vibration, 291, 525-538.

[13] E.Viola, M. Dilena and F. Tornabene, 2007. *Anlytical and numerical results for vibration analysis of multi-stepped and multi-damaged circular arches*, J. Sound and Vibration, 299, 143-163.

[14] C. H. Dönmez, 2012. *Theoritical and experimental analysis of vibration of curved beams with varrying cross-sections*, PhD. Thesis, Istanbul Technical University, Istanbul, (In Turkish).

[15] E. Tüfekci and A. Arpacı, 1998. *Exact solution of in-plane vibrations of circular arches with account taken of axial extension, transverse shear and rotatory inertia effects*, J. Sound and Vibration, 209(5), 845-856 .

[16] U. Eroğlu, 2014 . *Finite element analysis of curved beams using exact solutions*, MSc. Thesis, Istanbul Technical University, Istanbul, (In Turkish).

[17] E. Tüfekci and Ö. Özdemirci, 2006. E*xact solution of free in-plane vibration of a stepped circular arch*, J. Sound and Vibration, 295, 725-738.

[18] F. Durbin, 1974. *Numerical inversion of laplace transforms: an efficient improvement to Dubner and Abate's method*, Comput. J., 17,371 – 376

[19] B. Temel, 2004. *Transient analysis of viscoelastic helical rods subjectto time-dependent loads*, International J. of Solids and Structures, 41, 1605 –1624.

[20] ANSYS Swanson Analysis System, Inc., 201 Johnson Road, Houston, PA15342-1300,USA.