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Optimal Inequalities for Hemi-Slant Riemannian Submersions

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Abstract: In the present paper, we establish some basic inequalities involving the Ricci and scalar curvature of the vertical and the horizontal distributions for hemi-slant submersions having the total space a complex space form. We also discuss the equality case of the obtained inequalities and provide illustrative examples.

Keywords: Chen–Ricci inequality; Riemannian submersion; hemi-slant submersion; complex space form; Kähler manifold

MSC: 53C15; 53B20

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1. Introduction

As a dual notion to the isometric immersions, O'Neill and Gray introduced independently the concept of Riemannian submersions in [1,2], respectively. Riemannian submersions play an important role in mathematical and theoretical physics, especially due to their usage in the superstring, Yang–Mills, Kaluza–Klein and supergravity theories [3–8]. For more information on Riemannian submersions, we refer to the monographs [9,10].

In [11], Taştan, Şahin and Yanan introduced and investigated hemi-slant submersions from almost Hermitian manifolds onto Riemannian manifolds. This class of submersions appears as a natural generalization of invariant, anti-invariant, semi-invariant and slant submersions, four families of Riemannian submersions with remarkable geometric properties thoroughly investigated by Şahin [12–15]. Later, these submersions were studied for different ambient spaces by various authors who obtained several results regarding their geometry (see, e.g., [16–23]).

One of the most important curvature invariants for a Riemannian manifold (M, g) was introduced by Chen [24] as follows:

$$\delta_M = \tau(p) - \inf(K)(p), \tag{1}$$

where $\tau(p)$ is scalar curvature of M and

$$\inf(K)(p) = \inf\{K(\Pi) : \Pi \text{ is a plane section of } TpM\}. \tag{2}$$

In ref. [25], B.-Y. Chen established a general optimal inequality involving the intrinsic invariant δ_M and the squared mean curvature of a submanifold M isometrically immersed in a real space form $R(c)$ of constant sectional curvature c. This result gave rise to a

whole theory, known as the theory of Chen's invariants, which gained an exponential development in the following years (see the monograph [26] and the recent articles [27–32], as well as the references cited therein). The main purpose of this new theory is to prove answers to a fundamental problem in the geometry of submanifolds, namely "*establish simple relationships between the main extrinsic invariants and the main intrinsic invariants of a submanifold*" [26]. Recently, Chen-like inequalities have been investigated in the setting of Riemannian submersions (see, e.g., [19,33–36]).

Motivated by the studies indicated above, we obtain in this work various Chen-like inequalities for hemi-slant Riemannian submersions from complex space forms onto Riemannian manifolds and discuss the equality case of the obtained inequalities. The paper is organized as follows. In Section 2, we recall the definition and some fundamental properties of hemi-slant submersions. In Section 3, we derive the main inequalities: we first establish a Chen-like inequality involving the Ricci curvature and then state a Chen–Ricci inequality for the vertical and the horizontal distributions of hemi-slant Riemannian submersions with total space a complex space form, and with base an arbitrary Riemannian manifold. We also discuss the equality case of the obtained inequalities. In Section 4 , we provide examples of hemi-slant Riemannian submersions to show that the equality cases of the main inequalities can be attained.

2. Hemi-Slant Riemannian Submersions

In this study, manifolds, mappings, vector fields, sections, and so on, will always be supposed of class C^∞. We first recall the following definition.

Definition 1 ([11]). *Let (M, J, g) be an almost Hermitian manifold and (N, g_N) be a Riemannian manifold. A Riemannian submersion $\sigma : (M, J, g) \to (N, g_N)$ is said to be a hemi-slant submersion if there is a distribution $\mathcal{D}^\perp \subset \ker \sigma_*$ such that*

$$\ker \sigma_* = \mathcal{D}^\perp \oplus \mathcal{D}^\theta, \;\; J\left(\mathcal{D}^\perp\right) \subseteq (ker\sigma_*)^\perp,$$

and the angle $\theta = \theta(X)$ between JX and the space $\left(\mathcal{D}^\theta\right)_q$ is constant for nonzero $X \in \left(\mathcal{D}^\theta\right)_q$ and $q \in M$, where $\mathcal{D}^\theta$ is the orthogonal complement of $\mathcal{D}^\perp$ in $ker\sigma_*$. In this case, θ is called the hemi-slant angle of σ. Moreover, the hemi-slant submersion σ is called proper if $\mathcal{D}^\perp \neq \{0\}$ and $\theta \neq 0, \frac{\pi}{2}$.

Throughout this paper, we will assume all horizontal vector fields as basic vector fields. Let $\sigma : (M, g, J) \to (N, g_N)$ be a hemi-slant submersion. For $U \in ker\sigma_*$, we obtain

$$JU = \phi U + \omega U, \tag{3}$$

where $\phi U \in \ker \sigma_*$ and $\omega U \in (\ker \sigma_*)^\perp$. For $Z \in (\ker \sigma_*)^\perp$, we obtain

$$JZ = \mathcal{B}Z + \mathcal{C}Z \tag{4}$$

where $\mathcal{B}Z \in \ker \sigma_*$ and $\mathcal{C}Z \in (\ker \sigma_*)^\perp$. We have

$$(\ker \sigma_*)^\perp = J\mathcal{D}^\perp \oplus \omega\mathcal{D}^\theta \oplus \mu,$$

where μ is the orthogonal complement of $J\mathcal{D}^\perp \oplus \omega\mathcal{D}^\theta$ in $(ker\sigma_*)^\perp$ and is invariant under J. Let us consider the O'Neill's tensors $\mathcal{T}$ and $\mathcal{A}$ given by [1]

$$\mathcal{T}_\xi \eta = \mathfrak{v}\nabla_{\mathfrak{v}\xi}\mathfrak{h}\eta + \mathfrak{h}\nabla_{\mathfrak{v}\xi}\mathfrak{v}\eta, \;\; \mathcal{A}_\xi \eta = \mathfrak{v}\nabla_{\mathfrak{h}\xi}\mathfrak{h}\eta + \mathfrak{h}\nabla_{\mathfrak{h}\xi}\mathfrak{v}\eta \tag{5}$$

for any vector fields ξ and η on M, where $\mathfrak{v}$ and $\mathfrak{h}$ denote the vertical and horizontal projections of the submersion, and ∇ is the Levi-Civita connection of g. On the other hand, for any $X, Y \in \Gamma((ker\sigma_*)^{\perp})$ and $V, W \in \Gamma(ker\sigma_*)$, from (5), we obtain

$$\nabla_V W = \mathcal{T}_V W + \hat{\nabla}_V W, \tag{6}$$

$$\nabla_V X = \mathcal{T}_V X + \mathfrak{h}\nabla_V X, \tag{7}$$

$$\nabla_X V = \mathcal{A}_X V + \mathfrak{v}\nabla_X V, \tag{8}$$

$$\nabla_X Y = \mathfrak{h}\nabla_X Y + \mathcal{A}_X Y, \tag{9}$$

where $\hat{\nabla}_V W = \mathfrak{v}\nabla_V W$.

We denote by R, $R^{'}$, $\hat{R}$ and R^* the Riemannian curvature tensors of Riemannian manifolds M, N, the vertical distribution $ker\sigma_*$ and the horizontal distribution $(ker\sigma_*)^{\perp}$, respectively. Then, the Gauss–Codazzi type equations are given by [1]

$$R(U,V,F,W) = \hat{R}(U,V,F,W) + g(\mathcal{T}_U W, \mathcal{T}_V F) - g(\mathcal{T}_V W, \mathcal{T}_U F) \tag{10}$$

$$\begin{aligned} R(X,Y,Z,H) = {} & R^*(X,Y,Z,H) - 2g(\mathcal{A}_X Y, \mathcal{A}_Z H) \\ & + g(\mathcal{A}_Y Z, \mathcal{A}_X H) - g(\mathcal{A}_X Z, \mathcal{A}_Y H) \end{aligned} \tag{11}$$

$$\begin{aligned} R(X,V,Y,W) = {} & g((\nabla_X \mathcal{T})(V,W),Y) + g((\nabla_V \mathcal{A})(X,Y),W) \\ & - g(\mathcal{T}_V X, \mathcal{T}_W Y) + g(\mathcal{A}_Y W, \mathcal{A}_X V) \end{aligned} \tag{12}$$

where

$$\sigma_*(R^*(X,Y)Z)) = R^{'}(\sigma_* X, \sigma_* Y)\sigma_* Z \tag{13}$$

for all $U, V, F, W \in \Gamma(ker\sigma_*)$ and $X, Y, Z, H \in \Gamma((ker\sigma_*)^{\perp})$.

Moreover, the mean curvature vector field $\mathcal{H}$ of any fiber of Riemannian submersion σ is given by

$$\mathcal{H} = \frac{1}{r}\sum_{j=1}^{r} \mathcal{T}_{U_j} U_j, \tag{14}$$

where $\{U_1, \ldots, U_r\}$ is an orthonormal basis of the vertical distribution $ker\sigma_*$. Furthermore, σ has totally geodesic fibers if $\mathcal{T}$ vanishes identically.

Let M be an almost Hermitian manifold with an almost complex structure J and a Hermitian metric g. If J is parallel with respect to the Levi–Civita connection ∇ on M, that is

$$(\nabla_X J)Y = 0.$$

for all vector fields X and Y on M, then (M, J, g, ∇) is called a Kähler manifold. A complete and simply connected Kähler manifold M is said to be a complex space form if it has constant holomorphic sectional curvature c. In this case, the complex space form is denoted by $M(c)$. The curvature tensor of the complex space form $M(c)$ is given by

$$\begin{aligned} R(X,Y)Z = {} & \frac{c}{4}\{g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX \\ & - g(JX,Z)JY + 2g(X,JY)JZ\} \end{aligned} \tag{15}$$

for any $X, Y, Z \in \Gamma(TM)$.

The following theorem gives us a characterization of hemi-slant submersions (see [11]).

Theorem 1. *Let σ be a Riemannian submersion from an almost Hermitian manifold (M, g, J) onto a Riemannian manifold (N, g_N). Then, σ is a hemi-slant Riemannian submersion with hemi-slant angle θ if and only if there exist a distribution $\mathcal{D} \subset \ker \sigma_*$ and a constant $\lambda \in [0,1]$ such that*

(i) $\mathcal{D} = \{U \in \ker \sigma_* | \phi^2 U = -\lambda U\}$;

(ii) $\phi V = 0$, *for all* $V \in \mathcal{D}^{\perp}$, *where* $\mathcal{D}^{\perp}$ *is the orthogonal complement of* $\mathcal{D}$ *in* $\ker \sigma_*$.

Furthermore, we have $\cos^2 \theta = \lambda$.

By virtue of (3) and (4), we have the following result.

Lemma 1. *Let* $(M(c), g)$, (N, g_N) *be a complex space form and a Riemannian manifold, respectively. If* $\sigma : M(c) \to N$ *is a hemi-slant Riemannian submersion, then the following relations are valid*

$$g(\phi U, \phi V) = \cos^2 \theta g(U, V),$$
$$g(\omega U, \omega V) = \sin^2 \theta g(U, V),$$

for any $U, V \in \Gamma(\mathcal{D}^{\theta})$.

3. Chen–Ricci Inequality

In the present section, we aim to obtain some inequalities involving the Ricci curvature and the scalar curvature on the vertical and horizontal distributions for hemi-slant Riemannian submersions from a complex space form to a Riemannian manifold. We will also discuss the equality cases of these inequalities.

Let $\sigma : M(c) \to N$ be a proper hemi-slant Riemannian submersion from a complex space form $(M(c), g)$ onto a Riemannian manifold (N, g_N) and $\dim(\ker \sigma_*) = r = k_1 + 2k_2$. For every $q \in M$, we consider

$$\{U_1, \ldots, U_{k_1}, U_{k_1+1}, U_{k_1+2}, \ldots, U_{k_1+2k_2-1}, U_{k_1+2k_2}\}$$

an orthonormal basis of $(\ker \sigma_*)$ and $\{X_1 \ldots, X_n\}$ an orthonormal basis of $(\ker \sigma_*)^{\perp}$, respectively, such that $\{U_1, U_2, \ldots, U_{k_1}\}$ is an orthonormal basis of $D^{\perp}$, while

$$\{U_{k_1+1}, U_{k_1+2} = \sec\theta \phi U_{k_1+1}, \ldots, U_{k_1+2k_2-1}, U_{k_1+2k_2} = \sec\theta \phi U_{k_1+2k_2-1}\}$$

is an orthonormal basis of D^{θ}. We will call this basis an adapted hemi-slant basis of $(\ker \sigma_*)$. Obviously, we have

$$g^2(JU_i, U_{i+1}) = \begin{cases} 0, & \text{for } i \in \{1, \ldots, k_1 - 1\}, \\ cos^2\theta, & \text{for } i \in \{k_1 + 1, \ldots, k_1 + 2k_2 - 1\} \end{cases}$$

and

$$\sum_{i,j=1}^{r} g^2(JU_i, U_j) = 2k_2 cos^2\theta. \tag{16}$$

Besides from (10), (11) and (15), we have

$$\begin{aligned} \hat{R}(U, V, F, W) = \frac{c}{4}\{&g(V, F)g(U, W) - g(U, F)g(V, W) + g(U, JF)g(JV, W) \\ &- g(V, JF)g(JU, W) + 2g(U, JV)g(JF, W)\} \\ &- g(\mathcal{T}_U W, \mathcal{T}_V F) + g(\mathcal{T}_V W, \mathcal{T}_U F), \end{aligned} \tag{17}$$

for all vector fields $U, V, F, W \in \Gamma(\ker \sigma_*)$ and

$$\begin{aligned} R^*(X, Y, Z, H) = \frac{c}{4}\{&g(Y, Z)g(X, H) - g(X, Z)g(Y, H) + g(JY, Z)g(JX, H) \\ &- g(JX, Z)g(JY, H) + 2g(X, JY)g(JZ, H)\} + 2g(\mathcal{A}_X Y, \mathcal{A}_Z H) \\ &- g(\mathcal{A}_Y Z, \mathcal{A}_X H) + g(\mathcal{A}_X Z, \mathcal{A}_Y H) \end{aligned} \tag{18}$$

for all vector fields $X, Y, Z, H \in \Gamma((\ker \sigma_*)^{\perp})$.

Theorem 2. *Let $\sigma : M(c) \to N$ be a proper hemi-slant Riemannian submersion from a complex space form $(M(c), g)$ onto a Riemannian manifold (N, g_N). Then, we have*

$$\widehat{Ric}(U) \geq \frac{c}{4}(r - 1 + 3\cos^2\theta) - rg(\mathcal{T}_U U, \mathcal{H}) \tag{19}$$

for a unit vector field $U \in \Gamma(\mathcal{D}^\theta)$, where r is the dimension of the vertical distribution. The equality case of (19) holds identically for any unit vector field $U \in \Gamma(\mathcal{D}^\theta)$ if and only if each fiber is totally geodesic.

Proof. From (17), we obtain

$$\begin{aligned}\widehat{Ric}(U) = \frac{c}{4}\left[(r-1)g(U,U) + 3\sum_{i=1}^{r} g^2(U, JU_i)\right] \\ -rg(\mathcal{T}_U U, \mathcal{H}) + \sum_{i=1}^{r} \|\mathcal{T}_U U_i\|^2\end{aligned} \tag{20}$$

where

$$\widehat{Ric}(U) = \sum_{i=1}^{r} \hat{R}(U, U_i, U_i, U). \tag{21}$$

If $U \in \Gamma(\mathcal{D}^\theta)$, then choosing an adapted hemi-slant basis

$$\{U_1, \ldots, U_{k_1}, U_{k_1+1}, U_{k_1+2}, \ldots, U_{k_1+2k_2-1}, U_{k_1+2k_2}\}$$

of $\ker \sigma_*$, one derives

$$\sum_{i=1}^{r} g^2(U, JU_i) = cos^2\theta. \tag{22}$$

Using last equation in (20), we derive (19). On the other hand, it is clear that the equality case of (19) holds identically for any unit vector field $U \in \Gamma(\mathcal{D}^\theta)$ if and only if

$$\mathcal{T}_U U_i = 0, \ i = 1, \ldots, r$$

which means that the fibers are totally geodesic (see [9]). □

In a similar way, using an adapted hemi-slant basis of $\ker \sigma_*$, we obtain the following results.

Theorem 3. *Let $\sigma : M(c) \to N$ be a proper hemi-slant Riemannian submersion from a complex space form $(M(c), g)$ onto a Riemannian manifold (N, g_N). Then, we have*

$$\widehat{Ric}(U) \geq \frac{c}{4}(r - 1) - rg(\mathcal{T}_U U, \mathcal{H}) \tag{23}$$

for a unit vector field $U \in \Gamma(\mathcal{D}^\perp)$. The equality case of (23) holds identically for any unit vector field $U \in \Gamma(\mathcal{D}^\perp)$ if and only if each fiber is totally geodesic.

Theorem 4. *Let $\sigma : M(c) \to N$ be a proper hemi-slant Riemannian submersion from a complex space form $(M(c), g)$ onto a Riemannian manifold (N, g_N). Then, we have*

$$\widehat{Ric}(U, V) = \frac{c}{4}(r - 1 + 3\cos^2\theta)g(U, V) - rg(\mathcal{T}_U V, \mathcal{H}) + \sum_{i=1}^{r} g(\mathcal{T}_{U_i} V, \mathcal{T}_U U_i) \tag{24}$$

for $U, V \in \Gamma(\mathcal{D}^\theta)$.

Theorem 5. *Let $\sigma : M(c) \to N$ be a proper hemi-slant Riemannian submersion from a complex space form $(M(c), g)$ onto a Riemannian manifold (N, g_N). Then we have*

$$2\widehat{r} = \frac{c}{4}\{r^2 - r + 6k_2\cos^2\theta\} - r^2\|\mathcal{H}\|^2 + \sum_{i=1}^{r}\|\mathcal{T}_{U_i}U_i\|^2. \tag{25}$$

As a consequence of the last theorem, we derive the following.

Corollary 1. *Let $\pi : M(c) \to N$ be a proper hemi-slant Riemannian submersion from a complex space form $(M(c), g)$ onto a Riemannian manifold (N, g_N). Then, we have*

$$2\widehat{r} \geq \frac{c}{4}\{r^2 - r + 6k_2\cos^2\theta\} - r^2\|\mathcal{H}\|^2 \tag{26}$$

The equality case of (26) *holds if and only if each fiber is totally geodesic.*

Proof. The inequality (26) is clear from (25). On the other hand, the equality case of (26) holds if and only if

$$\mathcal{T}_{U_i}U_i = 0,\ i = 1, \ldots, r$$

which implies

$$\mathcal{T}_U U = 0, \tag{27}$$

for all $U \in \Gamma(\ker\sigma_*)$. Replacing U by $U + V$ in (27), where $U, V \in \Gamma(\ker\sigma_*)$, and using the symmetry of the O'Neill tensor $\mathcal{T}$ for vertical vector fields, we obtain $\mathcal{T}_U V = 0$. Hence, the fibers of the submersion are totally geodesic. □

Now, if $\{U_1, \ldots, U_r\}$ is an orthonormal basis of $\ker\sigma_*$ and $\{X_1 \ldots, X_n\}$ is an orthonormal basis of $(\ker\sigma_*)^\perp$, we denote

$$\mathcal{T}_{ij}^{s} = g(\mathcal{T}_{U_i}U_j, X_s), \tag{28}$$

where $1 \leq i, j \leq r$ and $1 \leq s \leq n$, and

$$\mathcal{A}_{ij}^{\alpha} = g(\mathcal{A}_{X_i}X_j, U_\alpha), \tag{29}$$

where $1 \leq i, j \leq n$ and $1 \leq \alpha \leq r$. From [35], we use

$$\delta(N) = \sum_{i=1}^{n}\sum_{k=1}^{r} g((\nabla_{X_i}\mathcal{T})_{U_k}U_k, X_i). \tag{30}$$

We also define

$$\|\mathcal{C}\|^2 = \sum_{i,j=1}^{n} g^2(\mathcal{C}X_i, X_j), \tag{31}$$

$$\|\mathcal{B}\|^2 = \sum_{i=1}^{n}\sum_{k=1}^{r} g^2(\mathcal{B}X_i, U_k) \tag{32}$$

and

$$\tau^* = \sum_{1\leq i<j\leq n} R^*(X_i, X_j, X_j, X_i). \tag{33}$$

Moreover, if $X \in \Gamma((\ker\sigma_*)^\perp)$, then

$$\|\mathcal{C}X\|^2 = \sum_{i=1}^{n} g^2(\mathcal{C}X, X_i) \tag{34}$$

and

$$Ric^*(X) = \sum_{i=1}^{n} R^*(X, X_i, X_i, X). \tag{35}$$

From the Binomial theorem, we have the following relation between the components of the O'Neill tensor field $\mathcal{T}$ and the squared mean curvature $\mathcal{H}$:

$$\sum_{s=1}^{n}\sum_{i,j=1}^{r}(\mathcal{T}_{ij}^{s})^2 = \frac{1}{2}r^2\|\mathcal{H}\|^2 + \frac{1}{2}\sum_{s=1}^{n}(\mathcal{T}_{11}^{s} - \mathcal{T}_{22}^{s} - \cdots - \mathcal{T}_{rr}^{s})^2 + 2\sum_{s=1}^{n}\sum_{j=2}^{r}(\mathcal{T}_{1j}^{s})^2 - 2\sum_{s=1}^{n}\sum_{2\le i<j\le r}\Big[\mathcal{T}_{ii}^{s}\mathcal{T}_{jj}^{s} - (\mathcal{T}_{ij}^{s})^2\Big]. \tag{36}$$

Theorem 6. *Let $\sigma : M(c) \to N$ be a proper hemi-slant Riemannian submersion from a complex space form $(M(c), g)$ onto a Riemannian manifold (N, g_N). Suppose U is a unit vertical vector field. Then:*

(i) If $U \in \Gamma(\mathcal{D}^{\perp})$, we have

$$\widehat{Ric}(U) \ge \frac{c}{4}(r-1) - \frac{1}{4}r^2\|\mathcal{H}\|^2. \tag{37}$$

(ii) If $U \in \Gamma(\mathcal{D}^{\theta})$, we have

$$\widehat{Ric}(U) \ge \frac{c}{4}(r-1+3\cos^2\theta) - \frac{1}{4}r^2\|\mathcal{H}\|^2. \tag{38}$$

Moreover, the equality cases of (37) and (38) hold if and only if there exist two orthonormal bases $\{U_1 = U, U_2, \ldots, U_r\}$ and $\{X_1 \ldots, X_n\}$ of $\ker\sigma_*$ and $(\ker\sigma_*)^{\perp}$, respectively, such that $\mathcal{T}_{12}^{\alpha} = \cdots = \mathcal{T}_{1r}^{\alpha} = 0$ and $\mathcal{T}_{11}^{\alpha} = \mathcal{T}_{22}^{\alpha} + \cdots + \mathcal{T}_{rr}^{\alpha}$, for $\alpha = 1, \ldots, n$.

Proof. Let $\{U_1, \ldots, U_{k_1}, U_{k_1+1}, U_{k_1+2}, \ldots, U_{k_1+2k_2-1}, U_{k_1+2k_2}\}$ be an adapted hemi-slant basis of $(\ker\sigma_*)$.
(i) Due to the fact that one can choose the above adapted hemi-slant basis such that $U_1 = U$, it suffices to prove (38) for $U = U_1$.

Using (28) in (25), we can write

$$2\widehat{r} = \frac{c}{4}\{r^2 - r + 6k_2\cos^2\theta\} - r^2\|\mathcal{H}\|^2 + \sum_{\alpha=1}^{n}\sum_{k,s=1}^{r}(\mathcal{T}_{ks}^{\alpha})^2. \tag{39}$$

If (36) is used in (39), then (39) can be rewritten as

$$2\widehat{r} = \frac{c}{4}\Big(r^2 - r + 6k_2\cos^2\theta\Big) - \frac{1}{2}r^2\|\mathcal{H}\|^2 + \frac{1}{2}\sum_{\alpha=1}^{n}(\mathcal{T}_{11}^{\alpha} - \mathcal{T}_{22}^{\alpha} - \cdots - \mathcal{T}_{rr}^{\alpha})^2 + 2\sum_{\alpha=1}^{n}\sum_{s=2}^{r}(\mathcal{T}_{1s}^{\alpha})^2 - 2\sum_{\alpha=p+1}^{b_1}\sum_{2\le k<s\le r}^{r}\Big[\mathcal{T}_{kk}^{\alpha}\mathcal{T}_{ss}^{\alpha} - (\mathcal{T}_{ks}^{\alpha})^2\Big]. \tag{40}$$

Thus, from (40) we derive

$$2\widehat{r} \ge \frac{c}{4}(r^2 - r + 6k_2\cos^2\theta) - \frac{1}{2}r^2\|\mathcal{H}\|^2 - 2\sum_{\alpha=1}^{n}\sum_{2\le i<j\le r}[\mathcal{T}_{ii}^{\alpha}\mathcal{T}_{jj}^{\alpha} - (\mathcal{T}_{ij}^{\alpha})^2]. \tag{41}$$

Furthermore, taking $U = W = U_i$, $V = F = U_j$ in (10), we obtain

$$2\sum_{2\le i<j\le r} R(U_i,U_j,U_j,U_i) = 2\sum_{2\le i<j\le r} \hat{R}(U_i,U_j,U_j,U_i) + 2\sum_{\alpha=1}^{n}\sum_{2\le i<j\le r}\left[\mathcal{T}_{ii}^{\alpha}\mathcal{T}_{jj}^{\alpha} - (\mathcal{T}_{ij}^{\alpha})^2\right]. \tag{42}$$

Using (42) in (41), we derive

$$2\widehat{r} \ge \frac{c}{4}\left(r^2 - r + 6k_2\cos^2\theta\right) - \frac{1}{2}r^2\|H\|^2 + 2\sum_{2\le k<s\le r}\hat{R}(U_k,U_s,U_s,U_k) - 2\sum_{2\le k<s\le r} R(U_k,U_s,U_s,U_k). \tag{43}$$

Furthermore, we have

$$2\widehat{r} = 2\sum_{2\le i<j\le r}\hat{R}(U_i,U_j,U_j,U_i) + 2\sum_{j=1}^{r}\hat{R}(U_1,U_j,U_j,U_1). \tag{44}$$

Considering (44) in (43), we derive

$$2\widehat{Ric}(U_1) \ge \frac{c}{4}\left(r^2 - r + 6k_2\cos^2\theta\right) - \frac{1}{2}r^2\|\mathcal{H}\|^2 - 2\sum_{2\le k<s\le r} R(U_k,U_s,U_s,U_k). \tag{45}$$

On the other hand, since $M(c)$ is a complex space form, its curvature tensor R satisfies (15) and we get

$$\sum_{2\le k<s\le r} R(U_k,U_s,U_s,U_k) = \frac{c}{4}\left[\frac{(r-2)(r-1)}{2} + 3\sum_{2\le k<s\le r} g^2(JU_k,U_s)\right]. \tag{46}$$

As $U_1 \in \Gamma(\mathcal{D}^{\perp})$, we obtain immediately

$$\sum_{2\le k<s\le r} g^2(JU_k,U_s) = k_2\cos^2\theta$$

and therefore (46) can be written as

$$\sum_{2\le k<s\le r} R(U_k,U_s,U_s,U_k) = \frac{c}{4}\left[\frac{(r-2)(r-1)}{2} + 3k_2\cos^2\theta\right] \tag{47}$$

Considering now the last equation in (45), we get

$$\widehat{Ric}(U_1) \ge \frac{c}{4}(r-1) - \frac{1}{4}r^2\|\mathcal{H}\|^2$$

and the conclusion is now clear.

(ii) Due to the fact that in this case one can choose the adapted hemi-slant basis

$$\{U_1,\ldots,U_{k_1},U_{k_1+1},U_{k_1+2},\ldots,U_{k_1+2k_2-1},U_{k_1+2k_2}\}$$

such that $U_{k_1+1} = U$, it suffices to prove (38) for $U = U_{k_1+1}$.

With similar arguments as in case (i), we obtain

$$2\widehat{Ric}(U_{k_1+1}) \ge \frac{c}{4}\{r^2 - r + 6k_2\cos^2\theta\} - \frac{1}{2}r^2\|\mathcal{H}\|^2 - 2\sum_{1\le k<s\le r;\ k,s\ne k_1+1} R(U_k,U_s,U_s,U_k). \tag{48}$$

and

$$\sum_{1\le k<s\le r;\ k,s\neq k_1+1} R(U_k,U_s,U_s,U_k) = \frac{c(r-2)(r-1)}{8} + \frac{3c}{4}\sum_{1\le k<s\le r;\ k,s\neq k_1+1} g^2(JU_k,U_s). \tag{49}$$

As $U_{k_1+1}\in\Gamma(\mathcal{D}^\perp)$, we obtain immediately

$$\sum_{1\le k<s\le r;\ k,s\neq k_1+1} g^2(JU_k,U_s) = (k_2-1)\cos^2\theta$$

and therefore (49) can be written as

$$\sum_{1\le k<s\le r;\ k,s\neq k_1+1} R(U_k,U_s,U_s,U_k) = \frac{c}{4}\left[\frac{(r-2)(r-1)}{2} + 3(k_2-1)\cos^2\theta\right]. \tag{50}$$

Considering now the last equation in (48), we get

$$\widehat{Ric}(U_{k_1+1}) \ge \frac{c}{4}(r-1+3\cos^2\theta) - \frac{1}{4}r^2\|\mathcal{H}\|^2$$

and inequality (38) is clear.

Now, we remark that the equality case of (37) holds if and only if the equality is attained in (41). However, this happens if and only if $\mathcal{T}^\alpha_{11} = \mathcal{T}^\alpha_{22} + \cdots + \mathcal{T}^\alpha_{rr}$ and $\mathcal{T}^\alpha_{1s} = 0$, for $s = 2,\ldots,r$ and $\alpha = 1,\ldots,n$. On the other hand, the equality case of (38) holds if and only, with respect to the hemi-slant adapted basis considered in the proof, we have $\mathcal{T}^\alpha_{k_1+1,k_1+1} = \sum_{i\neq k_1+1}\mathcal{T}^\alpha_{ii}$ and $\mathcal{T}^\alpha_{k_1+1,s} = 0$, for $\alpha = 1,\ldots,n$ and $s\in\{1\ldots,r\}-\{k_1+1\}$. Using a reordering of the vectors in the basis of $\ker\sigma_*$, we derive the conclusion. □

Theorem 7. *Let $\sigma : M(c)\to N$ be a proper hemi-slant Riemannian submersion from a complex space form $(M(c),g)$ onto a Riemannian manifold (N,g_N). Then, we have*

$$\tau^* \le \frac{c}{8}\left[n(n-1)+3\|\mathcal{C}\|^2\right]. \tag{51}$$

Moreover, the equality holds in (51) if and only if the horizontal distribution $(\ker\sigma_*)^\perp$ is integrable.

Proof. Using the anti-symmetry of $\mathcal{A}$ and (18), we obtain

$$2\tau^* = \frac{c}{4}\left[n(n-1)+3\sum_{i,j=1}^{n} g(\mathcal{C}X_i,X_j)g(\mathcal{C}X_i,X_j)\right] - 3\sum_{i,j=1}^{n} g(\mathcal{A}_{X_i}X_j,\mathcal{A}_{X_i}X_j), \tag{52}$$

where $\{X_1,\ldots,X_n\}$ is an orthonormal basis of $(\ker\sigma_*)^\perp$. Now, using (31) in (52) we obtain

$$2\tau^* = \frac{c}{4}\left[n(n-1)+3\|\mathcal{C}\|^2\right] - 3\sum_{i,j=1}^{n}\|\mathcal{A}_{X_i}X_j\|^2. \tag{53}$$

and inequality (51) follows immediately. Moreover, it is clear that the equality case of (51) holds if and only if $\mathcal{A}_{X_i}X_j = 0$, for $i,j = 1,\ldots,n$ and the proof is now complete due to the fact that the vanishing of the O'Neill tensor $\mathcal{A}$ is equivalent to the integrability of the horizontal distribution (see, e.g., [9]). □

Theorem 8. *Let $\sigma : M(c) \to N$ be a proper hemi-slant Riemannian submersion from a complex space form $(M(c), g)$ onto a Riemannian manifold (N, g_N). If $\{X_1, \ldots, X_n\}$ is an orthonormal basis of $(\ker \sigma_*)^{\perp}$, then we have*

$$Ric^*(X_1) = \frac{c}{4}\Big[(n-1) + 3\|\mathcal{C}X_1\|^2\Big] - 3\sum_{\alpha=1}^{r}\sum_{j=2}^{n}(\mathcal{A}_{1j}^{\alpha})^2. \tag{54}$$

Proof. By using (29) in (53), we have

$$2\tau^* = \frac{c}{4}\Big[n(n-1) + 3\|\mathcal{C}\|^2\Big] - 3\sum_{\alpha=1}^{r}\sum_{i,j=1}^{n}(\mathcal{A}_{ij}^{\alpha})^2. \tag{55}$$

Thus, (55) can be written as

$$2\tau^* = \frac{c}{4}\Big[n(n-1) + 3\|\mathcal{C}\|^2\Big] - 6\sum_{\alpha=1}^{r}\sum_{j=2}^{n}(\mathcal{A}_{1j}^{\alpha})^2 - 6\sum_{\alpha=1}^{r}\sum_{2\le i<j\le n}(\mathcal{A}_{ij}^{\alpha})^2. \tag{56}$$

Moreover, taking $X = H = X_i$, $Y = Z = X_j$ in (11), we obtain

$$\sum_{2\le i<j\le n} R(X_i, X_j, X_j, X_i) = \sum_{2\le i<j\le n} R^*(X_i, X_j, X_j, X_i) + 3\sum_{\alpha=1}^{r}\sum_{2\le i<j\le n}(\mathcal{A}_{ij}^{\alpha})^2. \tag{57}$$

Using (57) in (56), we derive

$$\begin{aligned} 2\tau^* = {} & \frac{c}{4}\Big[n(n-1) + 3\|\mathcal{C}\|^2\Big] - 6\sum_{\alpha=1}^{r}\sum_{j=2}^{n}(\mathcal{A}_{1j}^{\alpha})^2 \\ & + 2\sum_{2\le i<j\le n} R^*(X_i, X_j, X_j, X_i) - 2\sum_{2\le i<j\le n} R(X_i, X_j, X_j, X_i). \end{aligned} \tag{58}$$

Since $M(c)$ is a complex space form, its curvature tensor R satisfies the equality (15) and we obtain

$$\sum_{2\le i<j\le n} R(X_i, X_j, X_j, X_i) = \frac{c}{8}\left[(n-2)(n-1) + 6\sum_{2\le i<j\le n} g^2(\mathcal{C}X_i, X_j)\right]. \tag{59}$$

Then, from (58) and (59), taking into account that

$$\|\mathcal{C}\|^2 - 2\sum_{2\le i<j\le n} g^2(\mathcal{C}X_i, X_j) = 2\|\mathcal{C}X_1\|^2 \tag{60}$$

we get

$$2Ric^*(X_1) = \frac{c}{2}\Big[(n-1) + 3\|\mathcal{C}X_1\|^2\Big] - 6\sum_{\alpha=1}^{r}\sum_{j=2}^{n}(\mathcal{A}_{1j}^{\alpha})^2 \tag{61}$$

and equality (54) follows immediately. □

As an outcome of the above result, we have the following.

Theorem 9. *Let $\sigma : M(c) \to N$ be a proper hemi-slant Riemannian submersion from a complex space form $(M(c), g)$ onto a Riemannian manifold (N, g_N). If X is a unit horizontal vector field, then we have*

$$Ric^*(X) \le \frac{c}{4}\Big[(n-1) + 3\|\mathcal{C}X\|^2\Big]. \tag{62}$$

Moreover, the equality case of the above inequality holds identically for all unit horizontal vector fields if and only if the horizontal distribution is integrable.

Proof. Inequality (62) is clear from Theorem 8 because in (54) we can select $X_1 = X$ to be any arbitrary unit horizontal vector field. This is due to the fact that one can always choose in Theorem 8 an orthonormal basis $\{X_1, \ldots, X_n\}$ of $(\ker \sigma_*)^\perp$ with $X_1 = X$.

Now, if the horizontal distribution is integrable, then $\mathcal{A}_{X_i} X_j = 0$, for $i, j = 1, \ldots, n$, and it is clear that we have equality in (62). Conversely, if the equality case of (62) holds identically for all unit horizontal vector fields, then it follows that $\mathcal{A}^\alpha_{ij} = 0$, for $\alpha = 1, \ldots, r$ and $i, j = 1, \ldots, n$, $i \neq j$, which means $\mathcal{A}_{X_i} X_j = 0$, for all $i \neq j$. However, due to the skew-symmetry of $\mathcal{A}$ for horizontal vector fields, it is obvious that $\mathcal{A}_{X_i} X_i = 0$. Hence, $\mathcal{A}_{X_i} X_j = 0$ for $i, j = 1, \ldots, n$, and therefore the horizontal distribution is integrable. □

Now, we are going to state the Chen–Ricci inequality between the vertical and horizontal distributions for a hemi-slant Riemannian submersion $\sigma : M(c) \to N$ from a complex space form $(M(c), g)$ onto a Riemannian manifold (N, g_N). Suppose $\{U_1, U_2, \ldots, U_r\}$ is an orthonormal basis of $\ker \sigma_*$ and $\{X_1 \ldots, X_n\}$ is an orthonormal basis of $(\ker \sigma_*)^\perp$. Then, for the scalar curvature τ of $M(c)$, we have

$$2\tau = \sum_{k=1}^{r} Ric(U_k, U_k) + \sum_{s=1}^{n} Ric(X_s, X_s). \tag{63}$$

Further, we can write

$$\begin{aligned} 2\tau = &\sum_{j,k=1}^{r} R(U_j, U_k, U_k, U_j) + \sum_{i=1}^{n}\sum_{k=1}^{r} R(X_i, U_k, U_k, X_i) \\ &+ \sum_{i,s=1}^{n} R(X_i, X_s, X_s, X_i) + \sum_{s=1}^{n}\sum_{j=1}^{r} R(U_j, X_s, X_s, U_j). \end{aligned} \tag{64}$$

Next, let us denote as usual (see [35]):

$$\left\|\mathcal{T}^V\right\|^2 = \sum_{i=1}^{n}\sum_{k=1}^{r} g(\mathcal{T}_{U_k} X_i, \mathcal{T}_{U_k} X_i), \tag{65}$$

$$\left\|\mathcal{T}^H\right\|^2 = \sum_{k,j=1}^{r} g(\mathcal{T}_{U_k} U_j, \mathcal{T}_{U_k} U_j), \tag{66}$$

$$\left\|\mathcal{A}^V\right\|^2 = \sum_{i,j=1}^{n} g(\mathcal{A}_{X_i} X_j, \mathcal{A}_{X_i} X_j), \tag{67}$$

$$\left\|\mathcal{A}^H\right\|^2 = \sum_{i=1}^{n}\sum_{k=1}^{r} g(\mathcal{A}_{X_i} U_k, \mathcal{A}_{X_i} U_k). \tag{68}$$

Theorem 10. *Let $\sigma : M(c) \to N$ be a proper hemi-slant Riemannian submersion from a complex space form $(M(c), g)$ onto a Riemannian manifold (N, g_N). Suppose $\{U_1, U_2, \ldots, U_r\}$ is an orthonormal basis of $\ker \sigma_*$ and $\{X_1 \ldots, X_n\}$ is an orthonormal basis of $(\ker \sigma_*)^\perp$.*

(i) If $U_1 \in \Gamma(\mathcal{D}^\perp)$, then

$$\begin{aligned} &\frac{c}{2}\left[nr + n + r - 2 + 3\left(\|\mathcal{B}\|^2 + \|\mathcal{C} X_1\|^2\right)\right] \\ &\leq \widehat{Ric}(U_1) + Ric^*(X_1) + \frac{1}{4} r^2 \|\mathcal{H}\|^2 + 3\sum_{\alpha=1}^{r}\sum_{s=2}^{n} (\mathcal{A}^\alpha_{1s})^2 \\ &- \delta(N) + \left\|\mathcal{T}^V\right\|^2 - \left\|\mathcal{A}^H\right\|^2. \end{aligned} \tag{69}$$

(ii) If $U_1 \in \Gamma(\mathcal{D}^\theta)$, *then*

$$\begin{aligned}&\frac{c}{2}\Big[nr+n+r-2+3\Big(2\cos^2\theta+\|\mathcal{B}\|^2+\|\mathcal{C}X_1\|^2\Big)\Big]\\&\leq \widehat{Ric}(U_1)+Ric^*(X_1)+\frac{1}{4}r^2\|\mathcal{H}\|^2+3\sum_{\alpha=1}^{r}\sum_{s=2}^{n}(A_{1s}^{\alpha})^2\\&-\delta(N)+\Big\|\mathcal{T}^V\Big\|^2-\Big\|\mathcal{A}^H\Big\|^2.\end{aligned}\tag{70}$$

The equality case of (69) *and* (70) *holds if and only if*

$$\begin{aligned}\mathcal{T}_{11}^s&=\mathcal{T}_{22}^s+\cdots+\mathcal{T}_{rr}^s,\\\mathcal{T}_{1j}^s&=0,\end{aligned}$$

for $s=1,\ldots,n$, $j=2,\ldots,r$.

Proof. Since $M(c)$ is a complex space form, using (64) and (32) we get

$$2\tau=\frac{c}{4}[(n+r)(n+r-1)+6k_2\cos^2\theta+6\|\mathcal{B}\|^2+3\|\mathcal{C}\|^2].\tag{71}$$

On the other hand, using the Gauss–Codazzi type Equations (10)–(12), we derive

$$\begin{aligned}2\tau=&\,2\hat{\tau}+2\tau^*+r^2\|\mathcal{H}\|^2-\sum_{k,j=1}^{r}g(\mathcal{T}_{U_k}U_j,\mathcal{T}_{U_k}U_j)+3\sum_{i,s=1}^{n}g(\mathcal{A}_{X_i}X_s,\mathcal{A}_{X_i}X_s)\\&+\sum_{i=1}^{n}\sum_{k=1}^{r}\{g(\mathcal{T}_{U_k}X_i,\mathcal{T}_{U_k}X_i)-g(\mathcal{A}_{X_i}U_k,\mathcal{A}_{X_i}U_k)-g((\nabla_{X_i}\mathcal{T})_{U_k}U_k,X_i)\}\\&+\sum_{s=1}^{n}\sum_{j=1}^{r}\{g(\mathcal{T}_{U_j}X_s,\mathcal{T}_{U_j}X_s)-g(\mathcal{A}_{X_s}U_j,\mathcal{A}_{X_s}U_j)-g((\nabla_{X_s}\mathcal{T})_{U_j}U_j,X_s)\}.\end{aligned}\tag{72}$$

Therefore, using (30) and (36) in (72), we obtain

$$\begin{aligned}2\tau=&\,2\hat{\tau}+2\tau^*+\frac{1}{2}r^2\|\mathcal{H}\|^2-\frac{1}{2}\sum_{s=1}^{n}(\mathcal{T}_{11}^s-\mathcal{T}_{22}^s-\cdots-\mathcal{T}_{rr}^s)^2-2\sum_{s=1}^{n}\sum_{j=2}^{r}(\mathcal{T}_{1j}^s)^2\\&+2\sum_{s=1}^{n}\sum_{2\leq i<j\leq r}(\mathcal{T}_{ii}^s\mathcal{T}_{jj}^s-(\mathcal{T}_{ij}^s)^2)+6\sum_{\alpha=1}^{r}\sum_{s=2}^{n}(\mathcal{A}_{1s}^{\alpha})^2+6\sum_{\alpha=1}^{r}\sum_{2\leq i<s\leq n}(\mathcal{A}_{is}^{\alpha})^2\\&+\sum_{i=1}^{n}\sum_{k=1}^{r}\{g(\mathcal{T}_{U_k}X_i,\mathcal{T}_{U_k}X_i)-g(\mathcal{A}_{X_i}U_k,\mathcal{A}_{X_i}U_k)\}-2\delta(N)\\&+\sum_{s=1}^{n}\sum_{j=1}^{r}(g(\mathcal{T}_{U_j}X_s,\mathcal{T}_{U_j}X_s)-g(\mathcal{A}_{X_s}U_j,\mathcal{A}_{X_s}U_j))\end{aligned}\tag{73}$$

Using now (42), (57) and (71) in (73), we get

$$\begin{aligned}
&\frac{c}{4}[(n+r)(n+r-1)+6k_2\cos^2\theta+3\|\mathcal{C}\|^2+6\|\mathcal{B}\|^2]\\
&=2\widehat{Ric}(U_1)+2Ric^*(X_1)+\frac{1}{2}r^2\|\mathcal{H}\|^2\\
&-\frac{1}{2}\sum_{s=1}^{n}(\mathcal{T}_{11}^s-\mathcal{T}_{22}^s-\cdots-\mathcal{T}_{rr}^s)^2-2\sum_{s=1}^{n}\sum_{j=2}^{r}(\mathcal{T}_{1j}^s)^2\\
&+6\sum_{\alpha=1}^{r}\sum_{s=2}^{n}(\mathcal{A}_{1s}^\alpha)^2+\sum_{i=1}^{n}\sum_{k=1}^{r}\{g(\mathcal{T}_{U_k}X_i,\mathcal{T}_{U_k}X_i)-g(\mathcal{A}_{X_i}U_k,\mathcal{A}_{X_i}U_k)\}\\
&-2\delta(N)+\sum_{s=1}^{n}\sum_{j=1}^{r}\{g(\mathcal{T}_{U_j}X_s,\mathcal{T}_{U_j}X_s)-g(\mathcal{A}_{X_s}U_j,\mathcal{A}_{X_s}U_j)\}\\
&+2\sum_{2\le i<j\le r}R(U_i,U_j,U_j,U_i)+2\sum_{2\le i<j\le n}R(X_i,X_j,X_j,X_i).
\end{aligned}\tag{74}$$

Hence, in view of (65)–(68), the equality (74) implies

$$\begin{aligned}
&\frac{c}{4}[(n+r)(n+r-1)+6k_2\cos^2\theta+3\|\mathcal{C}\|^2+6\|\mathcal{B}\|^2]\\
&\le 2\widehat{Ric}(U_1)+2Ric^*(X_1)+\frac{1}{2}r^2\|\mathcal{H}\|^2\\
&+6\sum_{\alpha=1}^{r}\sum_{s=2}^{n}(\mathcal{A}_{1s}^\alpha)^2+2(\left\|\mathcal{T}^V\right\|^2-\left\|\mathcal{A}^H\right\|^2)-2\delta(N)\\
&+2\sum_{2\le i<j\le r}R(U_i,U_j,U_j,U_i)+2\sum_{2\le i<j\le n}R(X_i,X_j,X_j,X_i).
\end{aligned}\tag{75}$$

If $U_1\in\Gamma(\mathcal{D}^\perp)$, then considering (47) and (59) in (75), in view of (60) we obtain (69). Similarly, if $U_1\in\Gamma(\mathcal{D}^\theta)$, then using (50), (59) and (60) in (75), we obtain (70). Finally, the equality of (69) and (70) holds if and only if we have equality in (75), which happens if and only if $\mathcal{T}_{11}^s-\mathcal{T}_{22}^s-\cdots-\mathcal{T}_{rr}^s=0$ and $\mathcal{T}_{1j}^s=0$, for all $s=1,\ldots,n$ and $j=2,\ldots,r$. This completes the proof. □

Remark 1. *If $\sigma:M(c)\to N$ is a proper hemi-slant Riemannian submersion from a complex space form $(M(c),g)$ onto a Riemannian manifold (N,g_N), then from (71) and (72) we get*

$$\begin{aligned}
&\frac{c}{4}[(n+r)(n+r-1)+6k_2\cos^2\theta+3(\|\mathcal{C}\|^2+2\|\mathcal{B}\|^2)]\\
&=2\hat{\tau}+2\tau^*+r^2\|\mathcal{H}\|^2-\left\|\mathcal{T}^H\right\|^2+3\left\|\mathcal{A}^V\right\|^2\\
&-2\delta(N)+2\left\|\mathcal{T}^V\right\|^2-2\left\|\mathcal{A}^H\right\|^2.
\end{aligned}\tag{76}$$

From (76) we derive immediately that

$$\begin{aligned}
2\hat{\tau}+2\tau^*\le{}&\frac{c}{4}[(n+r)(n+r-1)+6k_2\cos^2\theta+3(\|\mathcal{C}\|^2+2\|\mathcal{B}\|^2)]\\
&-r^2\|\mathcal{H}\|^2+\left\|\mathcal{T}^H\right\|^2+2\delta(N)-2\left\|\mathcal{T}^V\right\|^2+2\left\|\mathcal{A}^H\right\|^2,
\end{aligned}\tag{77}$$

and

$$\begin{aligned}
2\hat{\tau}+2\tau^*\ge{}&\frac{c}{4}[(n+r)(n+r-1)+6k_2\cos^2\theta+3(\|\mathcal{C}\|^2+2\|\mathcal{B}\|^2)]\\
&-r^2\|\mathcal{H}\|^2+2\delta(N)-2\left\|\mathcal{T}^V\right\|^2+2\left\|\mathcal{A}^H\right\|^2-3\left\|\mathcal{A}^V\right\|^2.
\end{aligned}\tag{78}$$

Moreover, it is clear that the equality case of (77) holds for all $p \in M$ if and only if the horizontal distribution $(ker\sigma_*)^\perp$ is integrable, while the equality cases of (78) hold for all $p \in M$ if and only if the fibers of σ are totally geodesic submanifolds of $M(c)$. In particular, we deduce the following result.

Theorem 11. *Let $\sigma : M(c) \to N$ be proper a hemi-slant Riemannian submersion from a complex space form $(M(c), g)$ onto a Riemannian manifold (N, g_N) with totally geodesic fibers. Then, we have*

$$2\hat{\tau} + 2\tau^* \leq \frac{c}{4}[(n+r)(n+r-1) + 6k_2\cos^2\theta + 3(\|\mathcal{C}\|^2 + 2\|\mathcal{B}\|^2)] + 2\left\|\mathcal{A}^H\right\|^2. \tag{79}$$

Moreover, the equality case of (79) holds for all $p \in M$ if and only if the horizontal distribution $(ker\sigma_*)^\perp$ is integrable.

We now recall the following result, which we will use a little later.

Lemma 2 ([37])**.** *Let $a_1, a_2, \ldots, a_n$ be n real numbers $(n > 1)$. Then*

$$\frac{1}{n}\left(\sum_{i=1}^{n} a_i\right)^2 \leq \sum_{i=1}^{n} a_i^2$$

with equality iff $a_1 = a_2 = \ldots = a_n$.

Theorem 12. *Let $\sigma : M(c) \to N$ be a proper hemi-slant Riemannian submersion from a complex space form $(M(c), g)$ onto a Riemannian manifold (N, g_N). Then we have*

$$\begin{aligned}&\frac{c}{4}[(n+r)(n+r-1) + 6k_2\cos^2\theta + 3(\|\mathcal{C}\|^2 + 2\|\mathcal{B}\|^2)]\\ &\leq 2\hat{\tau} + 2\tau^* + r(r-1)\|\mathcal{H}\|^2 + 3\left\|\mathcal{A}^V\right\|^2 - 2\delta(N) + 2\left\|\mathcal{T}^V\right\|^2 - 2\left\|\mathcal{A}^H\right\|^2.\end{aligned} \tag{80}$$

Equality case of (80) holds for all $p \in M$ if and only σ has totally umbilical fibers.

Proof. From (76) we have

$$\begin{aligned}&\frac{c}{4}[(n+r)(n+r-1) + 6k_2\cos^2\theta + 3(\|\mathcal{C}\|^2 + 2\|\mathcal{B}\|^2)]\\ &= 2\hat{\tau} + 2\tau^* + r^2\|\mathcal{H}\|^2 - \sum_{s=1}^{n}\sum_{j=1}^{r}(\mathcal{T}_{jj}^s)^2 - \sum_{s=1}^{n}\sum_{j\neq k}^{r}(\mathcal{T}_{jk}^s)^2\\ &+ 3\left\|\mathcal{A}^V\right\|^2 - 2\delta(N) + 2\left\|\mathcal{T}^V\right\|^2 - 2\left\|\mathcal{A}^H\right\|^2.\end{aligned} \tag{81}$$

Applying Lemma 2 in (81), we get

$$\begin{aligned}&\frac{c}{4}[(n+r)(n+r-1) + 6k_2\cos^2\theta + 3(\|\mathcal{C}\|^2 + 2\|\mathcal{B}\|^2)]\\ &\leq 2\hat{\tau} + 2\tau^* + r^2\|\mathcal{H}\|^2 - \frac{1}{r}\sum_{s=1}^{n}(\sum_{j=1}^{r}\mathcal{T}_{jj}^s)^2 - \sum_{s=1}^{n}\sum_{j\neq k}^{r}(\mathcal{T}_{jk}^s)^2\\ &+ 3\left\|\mathcal{A}^V\right\|^2 - 2\delta(N) + 2\left\|\mathcal{T}^V\right\|^2 - 2\left\|\mathcal{A}^H\right\|^2,\end{aligned} \tag{82}$$

which gives (80). Equality case of (80) holds for all $p \in M$ if and only if the components of the O'Neill tensor $\mathcal{T}$ satisfy $\mathcal{T}_{11}^s = \mathcal{T}_{22}^s = \cdots = \mathcal{T}_{rr}^s$ and $\mathcal{T}_{jk}^s = 0$, for $s = 1, \ldots, n$, $j, k = 1, \ldots, r$, $j \neq k$. The conclusion is now clear. □

Using a similar proof as in Theorem 12, we deduce the following result.

Theorem 13. *Let $\sigma : M(c) \to N$ be a proper hemi-slant Riemannian submersion from a complex space form $(M(c), g)$ onto a Riemannian manifold (N, g_N). Then we have*

$$\frac{c}{4}[(n+r)(n+r-1) + 6k_2\cos^2\theta + 3(\|\mathcal{C}\|^2 + 2\|\mathcal{B}\|^2)] \geq 2\hat{\tau} + 2\tau^*$$
$$+ r^2\|\mathcal{H}\|^2 - \left\|\mathcal{T}^H\right\|^2 + \frac{3}{n}tr(\mathcal{A}^V)^2 - 2\delta(N) + 2\left\|\mathcal{T}^V\right\|^2 - 2\left\|\mathcal{A}^H\right\|^2 \tag{83}$$

The equality case of the above inequality holds for all $p \in M$ if and only if the components of the O'Neill tensor $\mathcal{A}$ with respect to some suitable orthonormal bases of the horizontal and vertical distributions satisfy $\mathcal{A}^s_{11} = \mathcal{A}^s_{22} = \cdots = \mathcal{A}^s_{nn}$ and $\mathcal{A}^s_{ij} = 0$, for $s = 1, \ldots, r$ and $i, j \in \{1, 2, \ldots, n\}$, $i \neq j$.

4. Examples

In this section, we provide examples of hemi-slant Riemannian submersions, illustrating the main results stated above.

From [11], we know that the concept of hemi-slant submersion generalizes in a natural way the notions of invariant, anti-invariant, semi-invariant and slant submersions. More precisely, if we denote the dimension of $\mathcal{D}^\perp$ and $\mathcal{D}^\theta$ by p_1 and p_2, respectively, then we have the following:

- If $\theta = 0$, then M is a semi-invariant submersion [12].
- If $\theta = 0$ and $p_1 = 0$, then M is an invariant submersion [15,38].
- If $\theta = 0$ and $p_2 = 0$, then M is an anti-invariant submersion [13].
- If $p_1 = 0$, then M is a slant submersion with slant angle θ [14].

We would like to point out that there is a special type of anti-invariant submersion, called Lagrangian submersion, for which the almost complex structure of the total space of the submersion reverses $\ker\sigma_*$ and $(\ker\sigma_*)^\perp$ (see [39]). Examples of invariant, anti-invariant, Lagrangian, semi-invariant, slant and hemi-slant submersions, as well as various interesting results regarding the geometry of these submersions, can be found in [11–15,39]. At this point, we would just like to note that according to Theorem 4.5 of [39], it follows that the horizontal distribution of a Lagrangian submersion with total space a complex space form is integrable. However, such submersions do not provide us suitable examples to illustrate the equality case of the inequalities stated in Theorems 7, 9 and 11, because a Lagrangian submersion is not a proper hemi-slant submersion.

Next, we will construct the first example of proper hemi-slant submersion satisfying the equality case of all inequalities established in the above section.

Example 1. *Consider the Kähler manifold $(\mathbb{R}^6, J, g_1)$ equipped with the canonical Euclidean metric g_1 and the complex structure J given by:*

$$J(x_1, x_2, x_3, x_4, x_5, x_6) = (-x_4, x_6, x_5, x_1, -x_3, -x_2).$$

Define now a map $\sigma : (\mathbb{R}^6, J, g_1) \to)(\mathbb{R}^3, g_2)$ by

$$\sigma(x_1, x_2, x_3, x_4, x_5, x_6) = (\frac{x_1 + x_4}{\sqrt{2}}, -\frac{-x_2 + x_5}{\sqrt{2}}, x_3\cos\alpha + x_6\sin\alpha),$$

where $\alpha \in (0, \frac{\pi}{2})$ and g_2 is the standard canonical Euclidean metric on $\mathbb{R}^3$. Then, it is easy to check that σ is a hemi-slant submersion such that

$$\mathcal{D}^\perp = Sp\{V_1 = -\frac{1}{\sqrt{2}}\frac{\partial}{\partial x_1} + \frac{1}{\sqrt{2}}\frac{\partial}{\partial x_4}\},$$

$$\mathcal{D}^{\theta} = Sp\{V_2 = -\frac{1}{\sqrt{2}}\frac{\partial}{\partial x_2} - \frac{1}{\sqrt{2}}\frac{\partial}{\partial x_5}, V_3 = -\sin\alpha\frac{\partial}{\partial x_3} + \cos\alpha\frac{\partial}{\partial x_6}\}$$

and

$$(ker\sigma_*)^{\perp} = Sp\{H_1 = \frac{1}{\sqrt{2}}\frac{\partial}{\partial x_1} + \frac{1}{\sqrt{2}}\frac{\partial}{\partial x_4}, H_2 = -\frac{1}{\sqrt{2}}\frac{\partial}{\partial x_2} + \frac{1}{\sqrt{2}}\frac{\partial}{\partial x_5},$$
$$H_3 = \cos\alpha\frac{\partial}{\partial x_3} + \sin\alpha\frac{\partial}{\partial x_6}\}.$$

Moreover, the hemi-slant angle of σ is $\theta = arccos(\frac{sin\alpha+cos\alpha}{\sqrt{2}})$. A straightforward computation shows that fibers of the submersions are totally geodesic and the horizontal distribution is integrable. Hence, we conclude that the inequalities stated in Theorems 2, 3, 6, 7, 9–11 and 13 are satisfied with equality sign.

Similarly, the following map illustrates the equality case of the above-mentioned inequalities.

Example 2. *Consider the Euclidean space $\mathbb{R}^{10}$ equipped with the standard metric (denoted by g_1) and the compatible almost complex structure J given by*

$$J(x_1, x_2, \ldots, x_9, x_{10}) = (x_7, x_3, -x_2, x_{10}, x_8, -x_9, -x_1, -x_5, x_6, -x_4).$$

Then, $(\mathbb{R}^{10}, J, g_1)$ is a Kähler manifold and we define a map $\sigma : (\mathbb{R}^{10}, J, g_1) \to (\mathbb{R}^5, g_2)$ by

$$\sigma(x_1, x_2, \ldots, x_{10}) = (t_1, t_2, t_3, t_4, t_5)$$

where

$$t_1 = x_5, \;\; t_2 = \sqrt{\frac{1}{1+7^{\sqrt{2}-1}}}x_1 + \sqrt{\frac{1}{1+7^{1-\sqrt{2}}}}x_7, \;\; t_3 = x_2$$
$$t_4 = x_4\tanh\alpha + x_{10}\text{sech}\alpha, \;\; t_5 = \frac{\sqrt{5}}{3}x_3 + \frac{2}{3}x_6,$$

$\alpha \in (0, \frac{\pi}{2})$ and g_2 is the standard canonical Euclidean metric on $\mathbb{R}^5$. A direct computation shows that σ is a hemi-slant submersion with the hemi-slant angle $\theta = \arccos(\frac{\sqrt{5}}{3})$ such that

$$\mathcal{D}^{\perp} = Sp\{V_1 = -\sqrt{\frac{1}{1+7^{1-\sqrt{2}}}}\frac{\partial}{\partial x_1} + \sqrt{\frac{1}{1+7^{\sqrt{2}-1}}}\frac{\partial}{\partial x_7},$$
$$V_2 = -\tanh\alpha\frac{\partial}{\partial x_{10}} + \text{sech}\alpha\frac{\partial}{\partial x_4}, V_3 = \frac{\partial}{\partial x_8}\},$$
$$\mathcal{D}^{\theta} = Sp\{V_4 = -\frac{2}{3}\frac{\partial}{\partial x_3} + \frac{\sqrt{5}}{3}\frac{\partial}{\partial x_6}, V_5 = \frac{\partial}{\partial x_9}\}.$$

and

$$(ker\sigma_*)^{\perp} = Sp\{H_1 = -\sqrt{\frac{1}{1+7^{1-\sqrt{2}}}}\frac{\partial}{\partial x_1} - \sqrt{\frac{1}{1+7^{\sqrt{2}-1}}}\frac{\partial}{\partial x_7},$$
$$H_2 = \tanh\alpha\frac{\partial}{\partial x_4} + \text{sech}\alpha\frac{\partial}{\partial x_{10}},$$
$$H_3 = -\frac{2}{3}\frac{\partial}{\partial x_3} - \frac{\sqrt{5}}{3}\frac{\partial}{\partial x_6}, H_4 = \frac{\partial}{\partial x_2}, H_5 = \frac{\partial}{\partial x_5}\}.$$

We derive immediately that fibers of the submersions are totally geodesic and the horizontal distribution is integrable. Hence, we have again that σ satisfies the equality case of the inequalities stated in Theorems 2, 3, 6, 7, 9–11 and 13.

5. Conclusions

(Semi-)Riemannian submersions are mathematical objects of high interest in theoretical physics (see, e.g., [9]). A particular class of such submersions was introduced by Taştan et al. [11], as a natural generalization of some important families of Riemannian submersions: invariant, anti-invariant, semi-invariant and slant submersions. In this paper, we prove various optimal inequalities involving basic curvature invariants for hemi-slant Riemannian submersions having as total space a complex space form and discuss the equality case of the obtained inequalities. Finally, we provide examples of hemi-slant Riemannian submersions to show that the equality cases of the main inequalities can be attained. For further research, it would be interesting to obtain Chen-like inequalities for lightlike submersions (see [40]). In this case, it could be necessary to use not only techniques from submanifold theory, but also from singularity theory (see, e.g., [41–44]).

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