Proc. Indian Acad. Sci. (Math. Sci.) (2019) 129:66
https://doi.org/10.1007/s12044-019-0502-z

Actions of some simple compact Lie groups on themselves

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MS received 7 September 2017; revised 10 December 2018; accepted 27 December 2018; published online 27 July 2019

Abstract. Let G be a compact connected simple Lie group acting non-transitively, non-trivially on itself. Hsiang (Cohomology theory of topological transformation groups, (1975) (New York: Springer)) conjectured that the principal isotropy subgroup type must be the maximal torus and the action must be cohomologically similar to the adjoint action and the orbit space must be a simplex. But Bredon (*Bull AMS* **83(4)** (1977) 711–718) found a simple counterexample, where the principal isotropy subgroup is not a maximal torus and which has no fixed point. In this work, we prove that if $SO(n)$, $(n \geq 34)$ or $SU(3)$ acts smoothly (and nontrivially) on itself with non-empty fixed point set, then the principal isotropy subgroups are maximal tori.

Keywords. Lie groups; differentiable transformation groups; characteristic classes.

2010 Mathematics Subject Classification. 57S15, 57S25.

1. Introduction

Given a Lie group and a differentiable manifold M, a (differentiable) map $\theta : G \times M \to M$ is called an action (of G on M) if it induces a group homomorphism from G to the group of diffeomorphisms of M. Usually, $\theta(g, x)$ is denoted by $g \cdot x$ (or gx) for short. When a group G acts on M, sometimes we will simply say M is a G-space. In the theory of transformation groups, one usually studies actions of the large (dimensional) groups (compared to the dimension of the manifold) because with some restrictions on the manifold (see, for example, [6,7] and [9]), there are very few possibilities for 'principal' orbits. On the other hand, when these dimensions are close (e.g. a Lie group acting on itself), there are many more possibilities. Given a Lie group, we already have three actions of G on itself: left (or right) action (which is transitive), the trivial action and the adjoint action. These actions are clearly distinct for non-abelian Lie groups. Hsiang, in [8], conjectured that these are 'essentially' the only possibilities for simple compact Lie groups. Hsiang's conjecture has been shown to be false by Bredon [2], who showed that $SU(3)$ action $\theta(A, B) = ABA^t$ on itself has no fixed point and the principal isotropy subgroups are not the maximal tori, so it is completely different from those three actions. For non simple Lie groups, there are other obvious actions formed by 'mixing' these three actions. For example, if G and H are Lie groups of positive dimension, $(g, h) \cdot (g', h') = (gg', hh'h^{-1})$ is a different

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action of $G \times H$ on itself. In this article, for some compact Lie groups acting on themselves, we exploit the cohomology ring structure of the group, to determine the principal orbit type when the fixed point set is nonempty. When studying actions of a group on a manifold, one should note that, given an action there are many 'similar' actions. Given (differentiable) action of $\theta : G \times M \to M$ of G on a manifold M, one can easily form new but similar actions of G on M; choose any diffeomorphism f of M and define θ' by $\theta'(g, x) = f^{-1}(\theta(g, f(x)))$. These two actions have exactly the same orbits, diffeomorphic fixed point sets and the homeomorphic orbit spaces, therefore indistinguishable. One naturally considers such actions as equivalent. Therefore, two actions with different principal isotropy orbits or non-diffeomorphic fixed point sets are not equivalent.

Our terminology is standard, the same as that used in [3].

2. Preliminaries

For (continuous) actions of a compact topological group G, the orbits of the action are (equivariantly) homeomorphic to homogeneous spaces G/H ($H = G_x$: isotropy subgroup of a point x in the orbit, therefore closed). The points in the same orbit have conjugate isotropy subgroups. The equivalence class of orbits equivalent to G/H is denoted by (G/H) (and the conjugacy class of closed subgroups H by (H)). Also, there is an equivariant map $G/H \to G/K$ iff K is conjugate to a subgroup of H. In this case, we write $(G/H) \geq (G/K)$ (and $(K) \leq (H)$), so this is a partial order in the set of equivalence classes of all orbits of G (respectively the set of conjugacy classes of all closed subgroups of G) [3].

A connected Lie group is called simple if its Lie algebra is simple [5]. Simple Lie algebras have been classified by Cartan and Killing. Each simple Lie group has a simply connected covering group which is also compact.

Theorem 1 (Differentiable slice theorem [10]). *Let G be a compact Lie group and M a G-manifold. For an arbitrary point x of M, the orbit $G(x)$ of x is a G-invariant submanifold of M. Let ν denote the normal G-vector bundle of $G(x)$ in M. Then the fibre ν_x over x of ν is a representation space of the isotropy group G_x so that ν is isomorphic to $G \times_{G_x} \nu_x \to G/G_x$ as smooth G-vector bundles. Moreover, there exist a G-invariant open neighborhood U of $G(x)$ in M and a G-diffeomorphism $f : G \times_{G_x} \nu_x \to U$ such that the restriction of f to the zero cross-section gives the G-diffeomorphism from G/G_x to $G(x)$. If we take a G-invariant Riemannian metric on M, then the G_x action on ν_x is given by the G_x-action on the orthogonal complement of $T_x(G(x))$ in $T_x(M)$.*

Theorem 2 (Principal orbit type theroem [3]). *For any differentiable G action on a connected manifold M, there exists a maximum orbit type G/H (called the principal orbit type and H is called the principal isotropy subgroup) (i.e., H is conjugate to a subgroup of each isotropy group). The union $M_{(H)}$ of the orbits of type G/H is open and dense in M, and its image $M^*_{(H)}$ in $M^* = M/G$ is connected.*

In this theorem G need not be simple, just compact. If x is a fixed point of the action, then principal isotropy subgroup type of φ_x (isotropy representation of G at T_xM) is the same as the principal isotropy subgroup type of G space action on M by the differentiable slice theorem [3].

Theorem 3 (Vietoris–Begle [11]). *Let X, Y be paracompact spaces, A be an abelian group and $f : X \to Y$ be a continuous closed surjection. If there is an integer $N \geq 0$ such that for each $0 \leq k \leq N$ and $\forall y \in Y$, $\widetilde{H}^k(f^{-1}(y); A) = 0$, then the homomorphism $\widetilde{H}^k(f) : \widetilde{H}^k(Y; A) \to \widetilde{H}^k(X; A)$ induced by f is*

(*i*) *an isomorphism for* $0 \leq k \leq N-1$;
(*ii*) *a monomorphism for* $k = N$.

Theorem 4 [4]. *Let $\pi : X \to Y$ be an n-sheeted covering space defined by a free action of a finite group on X. Then, with coefficients in a field F whose characteristic is 0 or a prime not dividing n, the map $\pi^* : H^k(Y; F) \to H^k(X; F)$ is injective.*

Theorem 5 ([9], Theorem I). *Let M^m be a G-space with orbits of $\mathbb{S}^{n-1}$-type and fixed points only. Then orbit space M/G is a manifold with boundary (of dimension $m-n+1$).*

Remark 6. Let M be a compact, connected, orientable (over a commutative ring R) manifold with (nonempty) boundary and $\dim M = n$. Then $H^k(M; R) \cong H_{n-k}(M, \partial M; R)$, $(\forall\, 0 \leq k \leq n)$ (Lefschetz duality). Therefore, $H^n(M; R) = 0$ if $\partial M \neq \emptyset$ [4].

Throughout this paper, we will assume that G is a simple Lie group, and every manifold and every action is smooth (of class C^∞). All actions of G on itself will be assumed to be nontrivial and nontransitive.

3. Adjoint-like actions

DEFINITION 7

Let G be a simple Lie group acting on itself. If

(i) the fixed point set is non-empty,
(ii) principal isotropy subgroups are maximal tori,

we will say that the action is *adjoint-like*.

It is clear that (since the isotropy representation at a fixed point has no trivial summand) adjoint-like actions have finitely many fixed points. For the adjoint action, the fixed point set is the center of the group.

DEFINITION 8

Let X be a G-space, Y be a H-space, $\Phi : G \to H$ a homomorphism and $f : X \to Y$ a map. Then f is a called Φ-equivariant, if $f(gx) = \Phi(g) f(x)$. If $G = H$, and Φ is identity map, we say f is equivariant.

DEFINITION 9

Let X, Y be two representation spaces (finite-dimensional real vector spaces on which each element of G acts linearly) of G. These representations are equivalent representations, if there is an equivariant (linear) isomorphism $f : X \to Y$.

Theorem 10. *Let G be a compact simple Lie group, $\tilde{G}$ its universal covering group (which is also compact). If every (nontrivial, nontransitive) action of $\tilde{G}$ on itself is adjoint-like, then every (nontrivial, nontransitive) action of G on itself is adjoint-like.*

Proof. Let $\theta : G \times G \to G$ be an (nontrivial, nontransitive) action. By [3], the action can be lifted to $\tilde{G}$ so that ($p : \tilde{G} \to G$ is the covering homomorphism)

$$\begin{array}{ccc} \tilde{G} \times \tilde{G} & \xrightarrow{\tilde{\theta}} & \tilde{G} \\ \downarrow{\scriptstyle p\times p} & & \downarrow{\scriptstyle p} \\ G \times G & \xrightarrow[\theta]{} & G \end{array}$$

is commutative and $\tilde{\theta}$ is also nontrivial and nontransitive. Then $\tilde{\theta}$ is an adjoint-like action. Let $\tilde{x}$ be a fixed point of $\tilde{G}$. Then $x = p(\tilde{x})$ is a fixed point of the G action. So (by Chapter I, Theorem 9.1 of [3]), we can assume that $\tilde{\theta}$ is an action of G on $\tilde{G}$ (i.e. elements of $\ker p$ act trivially on $\tilde{G}$).

Let $\varphi_{\tilde{x}}$ and φ_x be the isotropy representations of G at $\tilde{x}$ and x respectively. Since p is a covering map, it is a local diffeomorphism, so $dp : T_{\tilde{x}}\tilde{G} \to T_x G$ is isomorphism. Since p is equivariant, $dp : T_{\tilde{x}}\tilde{G} \to T_x G$ is an equivariant isomorphism. So $\varphi_{\tilde{x}}$ and φ_x are equivalent representations. Since principal isotropy subgroup type of $\varphi_{\tilde{x}}$ is maximal torus, the principal isotropy subgroup type of φ_x is also a maximal torus. So the action of G on itself is also adjoint-like. □

4. $SO(n)$ actions on itself

In this section, all cohomology groups will have coefficients in $\mathbb{Z}_2$ and will be suppressed. It is known [14] that $H^*(SO(n)) = \frac{\mathbb{Z}_2[e_1,e_3,\ldots,e_{2m-1}]}{(e_1^{a_1},e_3^{a_2},\ldots,e_{2m-1}^{a_m})}$, $m = \lfloor \frac{n}{2} \rfloor$, where a_i is the smallest power of 2 such that $a_i(2i-1) \geq n$.

Claim 11. In $H^*(SO(n))$, $e_1 e_2 \cdots e_{n-1} \neq 0$ (where $e_{2i} = e_i^2$ for $2i \leq n-1$).

Proof. $H^*(SO(n)) = \frac{\mathbb{Z}_2[e_1,e_3,\ldots,e_{2m-1}]}{(e_1^{a_1},e_3^{a_2},\ldots,e_{2m-1}^{a_m})}$, $m = \lfloor \frac{n}{2} \rfloor$, where a_i is the smallest power of 2 such that $a_i(2i-1) \geq n$.

For every $t \leq m = \lfloor \frac{n}{2} \rfloor$, let l be the largest integer such that $(2t-1)2^l \leq n-1$. Then $(2t-1)2^{l+1} \geq n$. Therefore, $a_t = 2^{l+1}$ and $2^{l+1} - 1 < a_t$, ${e_{2t-1}}^1 \cdot {e_{2t-1}}^2 \cdot {e_{2t-1}}^{2^2} \cdots {e_{2t-1}}^{2^l} = {e_{2t-1}}^{1+2+\cdots+2^l} = {e_{2t-1}}^{2^{l+1}-1}$. So $e_1 \cdot e_2 \cdots e_{n-1} \neq 0$. □

Let ρ_n (ρ for short) denote the standard representation, ϵ the 1-dimensional trivial representation and $\mathrm{Ad}_{SO(n)}$ (Ad for short) the adjoint representation of $SO(n)$.

Theorem 12. *Let $n \geq 17$, φ be an $\frac{1}{2}n(n-1)$-dimensional representation of $SO(n)$ with nontrivial principal isotropy subgroup type. Then $\varphi = a\epsilon + b\rho + c\,\mathrm{Ad}$, for dimensional*

reasons (*see* [7], *p.* 196). *Hence* (H_φ) (*the principal isotropy subgroup type of the representation* φ) *is*

$$(H_\varphi) = \begin{cases} (SO(n)) & b = c = 0, \\ (SO(n-b)) & c = 0,\ b > 0, \\ (T^{\lfloor \frac{n}{2} \rfloor}) & c = 1. \end{cases}$$

Theorem 13 [7]. *Let* φ *be a differentiable action of* $SO(n)$ *on a smooth manifold* M^m *with vanishing first Pontryagin class. If the identity component of the principal isotropy subgroup* H_φ^o *of* φ *is conjugate to a standard* $SO(l)$ *for* $l \geq 4$, *then we have* $G_x^o \sim SO(k_x)$ *for all* $x \in M^m$, *where* $SO(k_x)$ *is embedded in a standard manner in* $SO(n)$.

Theorem 14 [1]. $\widetilde{KO}(\mathbb{R}\mathrm{P}^m)$ *is* (*additively*) *a cyclic group of order* $2^{\lfloor \frac{m}{2} \rfloor}$ *if* $m \equiv 0, 3, 5, 6, 7 \bmod 8$, $2^{\lfloor \frac{m}{2} \rfloor + 1}$ *otherwise*, *generated by the equivalence class of* $\xi - \mathbf{1}$, *where* ξ *denotes the canonical line bundle and* $\mathbf{1}$ *the trivial line bundle.*

This theorem, together with the fact that $\tau(\mathbb{R}\mathrm{P}^m) \oplus \mathbf{1} = (m+1)\xi$, will be used to rule out some orbit types.

DEFINITION 15

Real Stiefel manifolds are the space of the orthonormal k-frames ($k \leq n$) in $\mathbb{R}^n$, denoted by $V_{n,k}$. $V_{n,k}$ is diffeomorphic to $SO(n)/SO(n-k)$ $\dim V_{n,k} = nk - \frac{1}{2}k(k+1)$. It is known that $V_{n,k}$ is $n-k-1$ connected and is parallelizable for $k \geq 2$ [12].

Claim 16. For all $1 < b \leq \lfloor \frac{n-1}{2} \rfloor$ $(n \geq 6)$, there exists (an integer) k, such that $k < n - b - 1 < 2k \leq n - 1$.

In p. 730, Remark 2.2 of [6], a claim similar to the following theorem has been made for all classical groups, without proof (with different restrictions on dimensions).

Theorem 17. *Let* $G = SO(n)$, $n \geq 34$ *act* (*differentiably*) *on an* $\frac{1}{2}n(n-1)$*-dimensional paralellizable space* M *with a* (*real*) *Stiefel manifold* $V_{n,k}$ $(1 < k < \frac{1}{2}(n-1))$ *as principal orbit. Then* G_x/G_x^0 (G_x^0 : *the connected component of the identity*) *has odd order for all* $x \in M$.

Proof. Suppose G_x/G_x^0 has even order for some $x \in M$. By Theorem 13, G_x^0 is a (embedded in a standard manner) $SO(m)$ for some m $(m > \frac{1}{2}n)$. Let H be a subgroup such that $SO(m) \subset H \subseteq G_x$ with $H/SO(m) = \mathbb{Z}_2$. Then there is an element $g \in H \setminus SO(m)$ such that $g^2 = e$ (identity). The pull-back of the tangent bundle of M under the map $G/H \to G/G_x \hookrightarrow M$ splits as $\tau(G/H) \oplus \eta$, where η is the vector bundle associated to the $\mathbb{Z}_2$ bundle $SO(n)/SO(m) \to SO(n)/H$ via a representation of $\mathbb{Z}_2$ (the restriction of the normal representation of G_x to $\mathbb{Z}_2$) and this representation is nontrivial, since the principal isotropy subgroups are connected. Since $SO(n)/SO(m)$ is $(m-1)$ connected, $SO(n)/SO(m) \to SO(n)/H$ is a $(m-1)$ universal $\mathbb{Z}_2$ principal bundle and $\mathbb{S}^{m-1} \to \mathbb{R}\mathrm{P}^{m-1}$ is a pull-back by some map $f : \mathbb{R}\mathrm{P}^{m-1} \to G/H$. Clearly, the pull-back

of the line bundle, associated to the sign representation of $\mathbb{Z}_2$ of G/H is the canonical line bundle ξ of the projective space. Since $k : n - m > 1$, the Stiefel manifold $V_{n,k} = SO(n)/SO(m)$ is paralellizable, and the tangent bundle of G/H is a sum of line bundles associated to the sign representation of $\mathbb{Z}_2$. So the pull-back of the tangent bundle of M to $\mathbb{R}\mathrm{P}^{m-1}$ is $l\xi$, $(\frac{1}{2}n(n-1) \geq l > 0)$ and it is trivial. But, by Theorem 14, we have $\frac{1}{2}n(n-1) \geq 2^{\lfloor \frac{m-1}{2} \rfloor}$ for some $m > \frac{1}{2}n$. A simple calculation shows that this is impossible for $n \geq 34$, a contradiction. □

Note. The claim is also valid for $k = 1$, with a similar, but easier argument. It is argued within the proof of the following theorem.

Theorem 18. *Let $n \geq 34$, $G = SO(n)$ act (nontrivially) on $M = G$ and x be a fixed point of the action. Let φ be the isotropy representation of $SO(n)$ at x. Then φ is (equivalent to) the adjoint representation.*

Proof. Let $\varphi = a\theta + b\rho + c\,\mathrm{Ad}$. Since the action is assumed to be nontrivial, φ can not be trivial. Therefore, one of b and c must be non-zero. We show $b = 0$ by contradiction. Assume that $b > 0$. We consider two cases:

Case I. $b = 1$. Then $(H_\varphi) = (SO(n-1))$, so principal isotropy subgroup of $SO(n)$ action on itself is $SO(n-1)$.

The principal orbit of the action is $SO(n)/SO(n-1) = \mathbb{S}^{n-1}$ (n − 1-dimensional sphere). $H = S(O(n-1) \times O(1))$ is the only subgroup with $SO(n-1) \subsetneq H \subsetneq SO(n)$. If there is a point y with isotropy subgroup $G_y = S(O(n-1) \times O(1))$, the orbit would be $\mathbb{R}\mathrm{P}^{n-1}$: $n-1$-dimensional real projective space. In that case, the normal bundle is the vector bundle associated with the representation of $S(O(n-1) \times O(1))/SO(n-1) \simeq \mathbb{Z}_2$. Therefore, $\tau(\mathbb{R}\mathrm{P}^{n-1}) \oplus k\xi \oplus l\mathbf{1}$ is the trivial bundle for for some k, l with $k + l = \frac{1}{2}n(n-1) - (n-1) = \frac{1}{2}(n-1)(n-2)$: codimension of this orbit. So, by Theorem 14, $(n + k) \geq 2^{\left\lfloor \frac{n-1}{2} \right\rfloor}$. But, for $n \geq 15$, this is impossible by a simple calculation (since $\frac{n-1}{2} \leq k \leq \frac{1}{2}(n-1)(n-2)$).

Therefore, the action has only two types of orbits: fixed points and the sphere $\mathbb{S}^{n-1}$. By Theorem 5, M/G is a manifold with (non-empty) boundary of dimension $\frac{(n-1)(n-2)}{2}$. We have $H^i(SO(n)/G_y) = 0$, $i < n-1$ for all orbits. By Vietoris–Begle theorem, $\pi^* : H^i(M/G) \to H^i(M)$ is an isomorphism for $i < n-1$. So, there are $f_i \in H^i(M/G)$ $(1 \leq i \leq n-2)$ such that $\pi^*(f_i) = e_i$, $i < n-1$. By Claim 11, $f_1 \cdots f_{n-2} \neq 0$ and deg $f_1 \cdots f_{n-2} = \frac{(n-1)(n-2)}{2}$. But by Remark 6, $H^{\frac{1}{2}(n-1)(n-2)}(M/G) = 0$, a contradiction.

Case II. $b > 1$. We have $SO(n)/G_y \simeq \frac{G/G_y^0}{G_y/G_y^0}$ $(\forall y \in G,\ G_y^0 = SO(k_y))$. By Theorem 17, G_y/G_y^0 has odd order for all $y \in X$. By Theorem 13, for all $y \in X$, $G_y^0 = SO(k_y)$, $n-b \leq k_y$ and $SO(n)/G_y^0$ is a Stiefel manifold. So by Theorem 4, we have $H^i(SO(n)/G_y) = 0$, $i < n-b$ for all orbits. By Vietoris–Begle theorem, $\pi^* : H^i(M/G) \to H^i(M)$ is an isomorphism for $i < n-b$. So, there are $f_i \in H^i(M/G)$ such that $\pi^*(f_i) = e_i$, $i < n-b$. By Claim 16, there is a k so that $k < n-b-1$ and $n-b-1 < 2k \leq n-1$. Since $e_k \in \mathrm{Im}\,\pi^*$ and π^* is ring homomorphism, $\pi^*(f_k^2) = e_k^2 = e_{2k}$. Since $e_1 \cdot e_2 \cdots e_{n-b-1}e_{2k} \neq 0$, we have $f_1 \cdot f_2 \cdots f_{n-b-1}f_k^2 \neq 0$. But since $\deg(f_1 f_2 \cdots f_{n-b-1}f_k^2) > \dim M/G = \frac{(n-b)(n-b-1)}{2}$, a contradiction.

Therefore, φ is the adjoint representation. □

COROLLARY 19

All (*nontrivial*, *nontransitive*) *actions of* $SO(n)$ (*for* $n \geq 34$) *on itself with nonempty fixed point set have maximal tori as principal isotropy subgroups* (*so the action is adjoint-like*).

5. $SU(3)$ actions on itself

In this section, all homology and cohomology groups will have coefficients in $\mathbb{Q}$ and will be suppressed.

It is known that [14] $H^*(SU(n)) = \Lambda_{\mathbb{Q}}(e_3, e_5, \ldots, e_{2n-1})$.

Theorem 20 [15]. *Let* φ *be an* 8*-dimensional non-trivial real representation of* $SU(3)$. *Let* (H_φ) *be the principal isotropy type of the action. Then there are two cases*:

(*i*) $\varphi = \mathrm{Ad}_{SU(3)}$, $(H_\varphi) = (T)$ *a maximal torus of* $SU(3)$,
(*ii*) $\varphi = \rho_3 +$ *trivial summand*, $(H_\varphi) = (SU(2))$, *where* $\rho_3 : SU(3) \to O(6)$ *is the standard representation.*

The following lemma shows that the top homology (and cohomology) of the orbit space is trivial for many differentiable actions of compact Lie groups on compact manifolds.

Lemma 21. *Let* X *be a compact, connected differentiable* G*-manifold, where* G *is a compact Lie group. Assume that there is a singular orbit such that all nearby orbits are principal. Then* $H_k(X/G) = 0$ $(k = \dim X/G)$ *for any coefficient ring.*

Proof. By [16], the orbit space is a (finite) polyhedron. We will prove that the k-th simplicial homology is zero. Let $c = \sum_{i=1}^{m} c_i \sigma_i$ be a k-dimensional simplicial cycle, where σ_i are the oriented k-simplices. After subdivision (if necessary), one $(k-1)$-dimensional face (consisting of singular orbits only) of at least one k-dimensional simplex is a face to only one k-dimensional simplex, say σ_1. Since c is a cycle, we have $c_1 = 0$. If X/G has no other k simplices, we are done. If not, by Theorem 2, X/G has an open, connected, dense subset, say U (the image of the principal orbits) which is a k-dimensional manifold. Each k-simplex σ_i, $i > 1$ has a common interior point with U. Choose one such point for each k simplex. Consider paths (in U) connecting that point to an arbitrary fixed interior point of σ_1 such that it crosses a $(k-1)$-dimensional face (transversally) while leaving or entering a k-simplex. We use induction on the (minimum) number of k-simplices these paths pass through. By Corollary 3 of [13], $(k-1)$-dimensional faces those paths cross belong to exactly two k-simplices. By induction hypothesis, the cycle c has 0 coefficient at one of these k simplices, so the other one also must have 0 coefficient. Thus $c = 0$, by induction. Hence $H_k(X/G) = 0$. □

Since we use a field as coefficients, we have $H^k(X/G) = 0$ by duality. Actually, a similar argument could also be used to show $H^k(X/G) = 0$ for any coefficient ring.

Theorem 22. *Let* $SU(3)$ *act on itself* (*nontrivially*) *with nonempty fixed point set. Then the principal isotropy subgroups are maximal tori* (*so the action is adjoint-like*).

Proof. Let φ be the isotropy representation of $SU(3)$ at a fixed point x. φ can not be trivial. Let (H_φ) be a principal isotropy type of φ. Suppose $\varphi = \rho_3 +$ trivial. Then the principal isotropy subgroup of $SU(3)$ action on itself is $SU(2)$.

The principal orbit of the action is $SU(3)/SU(2) \cong \mathbb{S}^5$. Also, we have only principal orbits near the fixed point x.

Let us show that there are no points with isotropy subgroup $S(U(2) \times U(1))$ (therefore, there are no orbits of type $\mathbb{C}\mathrm{P}^2 \simeq SU(3)/S(U(2) \times U(1))$). Suppose such an orbit exists. Let $j : \mathbb{C}\mathrm{P}^2 \simeq SU(3)/S(U(2) \times U(1) \hookrightarrow SU(3) = G$ the orbit inclusion and let ν be the normal bundle so that $8\epsilon = j^*(\tau G) = \tau\mathbb{C}\mathrm{P}^2 \oplus \nu$ (ϵ: the trivial line bundle). The bundle ν is isomorphic to the bundle associated to a representation of the isotropy subgroup $S(U(2) \times U(1))$ (by the slice theorem). The action of $SU(2)$ is trivial on ν since $SU(2)$ is the principal isotropy subgroup. So it arises from a representation of $\mathbb{S}^1$ via $S(U(2) \times U(1)) \to S(U(2) \times U(1))/SU(2) \simeq \mathbb{S}^1$. In particular, ν is a sum of certain complex line bundles, say ν_1, ν_2. Let $c_1(\nu_1) = k_1 x$ be the first Chern classes of ν_i, where $x \in H^2(\mathbb{C}\mathrm{P}^2)$ is a generator. Then the first Pontryagin class of ν is $p_1(\nu) = (k_1^2 + k_2^2)x^2$. On the other hand, $p_1(\mathbb{C}\mathrm{P}^2) = 3x^2$. So, we get $0 = p_1(j^*(8\epsilon)) = p_1(\tau\mathbb{C}\mathrm{P}^2 \oplus \nu) = p_1(\mathbb{C}\mathrm{P}^2) + p_1(\nu) = (3 + k_1^2 + k_2^2)x^2 \neq 0$, a contradiction. Therefore, with $G = SU(3) = X$ (for all non-fixed points), we have $G_y^0 = SU(2)$, G_y/G_y^0 is finite and $G/G_y = \frac{G/G_y^0}{G_y/G_y^0}$. Since $H^i(G/G_y^0) = 0$, $(i < 5)$ by Theorem 4, we have $H^i(Gy) = 0, i < 5$ for all orbits and by the Vietoris–Begle theorem, $\pi^* : H^k(X/G) \to H^k(X)$ is an isomorphism for $k < 5$. Therefore, $e_3 \in \operatorname{Im}\pi^*$. So, $H^3(X/G) \neq 0$. This contradicts Lemma 21. Hence, the principal isotropy subgroups of $SU(3)$ are maximal tori, so the action is adjoint-like. □

Acknowledgements

This study was financially supported by Tübitak-Bideb 2211 and Çukurova University (Project No. FDK-2015-4603). The authors would like to thank the anonymous referee for pointing out some errors and gaps in their arguments and also for suggesting improvements, especially in the proof of non-existence of orbits of type $\mathbb{C}\mathrm{P}^2$ in the last theorem.

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COMMUNICATING EDITOR: Parameswaran Sankaran