Ricerche di Matematica
https://doi.org/10.1007/s11587-026-01120-w

On symmetric polynomials of metabelian Lie algebras

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Received: 26 September 2025 / Accepted: 22 April 2026
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Abstract
Let F_n be the free metabelian Lie algebra of rank n over a field K of characteristic zero. In this work, we provide an explicit set of generators and a presentation of generators of the algebra $(F_n')^{S_n} = F_n^{S_n} \cap F_n'$ of symmetric polynomials as a right $K[X_n]^{S_n}$-module. Besides, we compute the Hilbert series of the K-space $(F_n')^{S_n}$.

Keywords Invariants · Metabelian · Symmetric

Mathematics Subject Classification 17B01 · 17B30 · 05E05

1 Introduction

Let K be a field of characteristic zero, $K[X_n] = K[x_1, \ldots, x_n]$ be the polynomial algebra on n commuting variables, $\mathfrak{V}$ be a variety of algebras over the ground field K, $F_n = F_n(\mathfrak{V})$ be the free algebra of rank n generated by X_n in the variety $\mathfrak{V}$. In this respect, $K[X_n]$ is the free algebra of rank n in the variety of commutative associative unital algebras. Hence, an element in $K[X_n]$ and F_n is a polynomial of variables $x_1, \ldots, x_n$ by the universal property of these free algebras. A polynomial $p(X_n) \in K[X_n]$ ($p(X_n) \in F_n$) is symmetric if

$$p(x_1, \ldots, x_n) = p(x_{\pi(1)}, \ldots, x_{\pi(n)})$$

for all $\pi \in S_n$. In other words, a polynomial is symmetric if it is preserved under every permutation of generators. The set of all symmetric polynomials of $K[X_n]$ (respectively F_n) forms a subalgebra that is called the algebra of invariants of S_n denoted by

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Published online: 21 May 2026

$K[X_n]^{S_n}$ (respectively $F_n^{S_n}$). It is well known that the algebra $K[X_n]^{S_n}$ is generated by the following algebraically independent elementary symmetric polynomials.

$$
\begin{aligned}
e_1 &= x_1 + \cdots + x_n \\
e_2 &= x_1x_2 + \cdots + x_1x_n + x_2x_3 + \cdots + x_{n-1}x_n \\
e_3 &= x_1x_2x_3 + \cdots + x_1x_2x_n + x_2x_3x_4 + \cdots + x_{n-2}x_{n-1}x_n \\
&\cdots \\
e_{n-1} &= x_1x_2\cdots x_{n-2}x_{n-1} + x_1x_2\cdots x_{n-2}x_n + \cdots + x_2x_3\cdots x_{n-1}x_n \\
e_n &= x_1x_2\cdots x_{n-1}x_n
\end{aligned}
$$

The algebra $F_n^{S_n}$ of symmetric polynomials were studied in many papers appeared recently. In the sequel, we list some of them considered in free and relatively free algebras with polynomial identities.

$*$ Free associative algebras. Initially Wolf [15] worked on the symmetric polynomials of the free associative algebras. Recently, in [3] the authors provided a generating set for the symmetric polynomials of F_n using Koryukin's action given in [12], which allows to decrease number of generators effectively, defined by an equivalence class determined by an additional action of symmetric group on the right side. With these two actions, the subalgebra of symmetric polynomials is generated by elements of the form

$$p_i = \sum_{\sigma \in S_n} x_{\sigma(1)} \cdots x_{\sigma(i)}, \quad i = 1, \ldots, n.$$

$*$ Relatively free associative algebras with the polynomial identity (PI): $[x, y][z, t] = 0$. In this case F_2 is the free algebra of rank 2 in the variety of metabelian associative algebras. In [8], the second named author found a finite generating set of $F_2^{S_2}$:

$$
\begin{gathered}
x_1 + x_2\,,\ x_1x_2 + x_2x_1\,,\ (x_2 - x_1)[x_2, x_1]\,,\ [x_2, x_1](x_2 - x_1)\,, \\
x_1(x_2 - x_1)[x_2, x_1]x_2 + x_2(x_2 - x_1)[x_2, x_1]x_1\,, \\
x_1[x_2, x_1](x_2 - x_1)x_2 + x_2[x_2, x_1](x_2 - x_1)x_1.
\end{gathered}
$$

$*$ Relatively free associative algebra with the PI: $[[x, y], z] = 0$. It was showed in [1] if, then $(F_2')^{S_2}$ is the free left $K[X_2]^{S_2}$-module generated by $(x_2 - x_1)[x_2, x_1]$, and that $(F_3')^{S_3}$ is a free $K[X_3]^{S_3}$ module of rank three, where F_3 is the free algebra in the variety generated by the infinite dimensional Grassmann algebra, where the $K[X_3]^{S_3}$-free generators are

$$
\begin{aligned}
&(x_2 - x_1)[x_2, x_1] + (x_3 - x_1)[x_3, x_1] + (x_3 - x_2)[x_3, x_2]\,, \\
&(x_2^2 - x_1^2)[x_2, x_1] + (x_3^2 - x_1^2)[x_3, x_1] + (x_3^2 - x_2^2)[x_3, x_2]\,, \\
&(x_1x_2^2 - x_1^2x_2)[x_2, x_1] + (x_1x_3^2 - x_1^2x_3)[x_3, x_1] + (x_2x_3^2 - x_2^2x_3)[x_3, x_2]\,.
\end{aligned}
$$

$*$ Relatively free associative algebras with the PI: $[x^2, y] = 0$. The authors proved in [9] that the algebra $F_2^{S_2}$ is the free module of $(\mathrm{Cent}F_2)^{S_2}$ with 4 generators. Here, F_2

stands for the associative algebra generated by two generic traceless matrices, and the four generators are the unity together with

$$x_1 + x_2, \quad x_2^2 x_1 + x_1^2 x_2, \quad (x_1^2 - x_2^2)[x_1, x_2].$$

Note that the Lie subalgebra of F_2 is the free algebra of rank 2 in the variety generated by the Lie algebra $sl_2(K)$.

$*$ Relatively free associative algebras with the PI: $(xy)z = (xz)y$; $x(yz) = y(xz)$. The algebra of symmetric polynomials was described [5] in the free bicommutative algebra F_n of rank n. As a special case, let $n = 2$. Then, $(F_2')^{S_2}$ is generated by

$$(y_1 + y_2)(z_1 + z_2)\,, \quad (y_1 + y_2)z_1 z_2\,, \quad y_1 y_2 (z_1 + z_2)\,, \quad y_1 y_2 z_1 z_2$$

as a module of $K[y_1 + y_2, y_1 y_2]K[z_1 + z_2, z_1 z_2]$, where y_i and z_j's stand for the left and the right actions of x_i and x_j's on elements of F_2', respectively.

We also mention the works considered in noncommutative or nonassociative algebras.

$*$ Free metabelian Lie algebras. An infinite minimal generating set of the Lie algebra $F_2^{S_2}$ were given in [10], while a set of generators of $(F_n')^{S_n}$, $n \geq 2$, was described in [6] as a right $K[X_n]^{S_n}$-module. In particular, if $n = 2$, then $(F_2')^{S_2}$ is the free $K[X_2]^{S_2}$-module generated by the single element $[x_2, x_1, x_2 - x_1]$. When $n = 3$, then the $K[X_3]^{S_3}$-module $(F_3')^{S_3}$ is (not freely) generated by

$$\begin{aligned}
&[x_2, x_1, x_2 - x_1] + [x_3, x_1, x_3 - x_1] + [x_3, x_2, x_3 - x_2],\\
&[x_2, x_1, x_2 - x_1, x_3] + [x_3, x_1, x_3 - x_1, x_2] + [x_3, x_2, x_3 - x_2, x_1],\\
&[x_2, x_1, x_2 - x_1, x_3, x_3] + [x_3, x_1, x_3 - x_1, x_2, x_2] + [x_3, x_2, x_3 - x_2, x_1, x_1].
\end{aligned}$$

$*$ Free metabelian Leibniz algebras. The description of symmetric polynomials and their generating sets in free Leibniz algebra F_n were provided in [11, 13]. In the case of rank 2, the $K[X_2]^{S_2}$-module $(F_2')^{S_2}$ is freely generated by the following four elements.

$$\begin{aligned}
&[x_1, x_1] + [x_2, x_2]\,, \quad [x_1, x_2] + [x_2, x_1]\,,\\
&[[x_1, x_1], x_1] + [[x_2, x_2], x_2]\,, \quad [[x_1, x_2], x_1] + [[x_2, x_1], x_2]\,.
\end{aligned}$$

$*$ Free metabelian Poisson algebras. A generating set of symmetric polynomials in the free Poisson algebra F_n were given in [7]. Recall that F_n is a Lie algebra with the commutator "$[.,.]$", and is an associative commutative algebra with the product "$\cdot$". The $K[X_2]^{S_2}$-module $(F_2')^{S_2}$ is generated by

$$\begin{aligned}
&[x_2, x_1](y_2 - y_1)\,, \quad x_1 \cdot x_1 + x_2 \cdot x_2\,, \quad x_1 \cdot x_2\\
&x_1 \cdot x_1 \cdot x_1 + x_2 \cdot x_2 \cdot x_2\,, \quad x_1 \cdot x_1 \cdot x_2 + x_1 \cdot x_2 \cdot x_2\\
&[x_2, x_1] \cdot (x_2 - x_1)\,, \quad [x_2, x_1, x_1] \cdot x_2
\end{aligned}$$

In the present study, we consider the free metabelian Lie algebra F_n, and give a presentation of generators of the algebra $F_n^{S_n}$ of symmetric polynomials for $n \geq 3$. Additionally, we compute the Hilbert series of the graded K-space $(F_n')^{S_n}$.

2 Preliminaries

A Lie algebra L is a vector space with a bilinear product $[.,.]$ called the *commutator* satisfying the following identities.

$$[x, x] = 0\,, \quad [[x, y], z] + [[y, z], x] + [[z, x], y] = 0.$$

If the characteristic of the ground field is different from 2, then the first identity is equivalent to $[x, y] + [y, x] = 0$ that is called the *skew symmetry*. The latter identity is called *Jacobi identity*. A Lie algebra is abelian if $[x, y] = 0$; and it is metabelian, if $[[x, y], [z, t]] = 0$ for all $x, y, z, t \in L$.

Let $\mathfrak{A}^2$ be the variety of metabelian Lie algebras over a fixed field K of characteristic zero. Now, we denote by $F_n = F_n(\mathfrak{A}^2)$ the free algebra of rank $n \geq 2$ generated by $X_n = \{x_1, \ldots, x_n\}$ in the variety $\mathfrak{A}^2$.

In the sequel, we use left normed commutators; that is,

$$[x, y, z] = [[x, y], z] = [x, y]\mathrm{ad}z$$

where $\mathrm{ad}z$ is the adjoint operator acting on the right side. The algebra F_n has a K-linear basis [2, 4] consisting of elements of the form

$$[x_i, x_j, x_{j_1}, \ldots, x_{j_k}]\,, \quad i > j \leq j_1 \leq \cdots \leq j_k.$$

Let u be an element in the commutator ideal F_n' of F_n generated by elements of the form $[x, y]$. Then, clearly $[x, y, u] = 0$ and

$$0 = [x, y, u] + [y, u, x] + [u, x, y] = -[u, y, x] + [u, x, y].$$

Thus $[u, x, y] = u\mathrm{ad}x\mathrm{ad}y = u\mathrm{ad}y\mathrm{ad}x = [u, y, x]$ by the above equality. Consequently, F_n' is a $K[X_n]$-module by the action

$$up(X_n) = up(\mathrm{ad}(X_n)) = up(\mathrm{ad}x_1, \ldots, \mathrm{ad}x_n)$$

and in particular, $ux_i = [u, x_i] = u\mathrm{ad}x_i$, where $u \in F_n'$.

2.1 The wreath product

We use same notations as in [6] for the embedding of the free metabelian Lie algebra F_n in an abelian wreath product of Lie algebras suggested by Shmel'kin [14]. Let

$$KU_n = \mathrm{Span}_K U_n = K\{u_1, \ldots, u_n\}$$

$$KT_n = \mathrm{Span}_K T_n = K\{t_1, \ldots, t_n\}$$

be the abelian Lie algebras with bases U_n and T_n, respectively. We set

$$W_n = u_1 K[X_n] \oplus \cdots \oplus u_n K[X_n]$$

that is the free right $K[X_n]$-module with free generating set U_n, and we assume that W_n a Lie algebra with trivial multiplication. The semidirect sum $W_n \leftthreetimes KT_n$ with elements of the form

$$w = \sum_{i=1}^{n} u_i p_i(X_n) + \sum_{i=1}^{n} \alpha_i t_i, \quad p_i(X_n) \in K[X_n], \alpha_i \in K. \tag{1}$$

is equal to the abelian wreath product $(KU_n)\mathrm{wr}(KT_n)$, in which the multiplication defined by

$$[W_n, W_n] = [T_n, T_n] = 0, \quad [u_i p_i(X_n), t_j] = u_i p_i(X_n) x_j,$$

where $i, j = 1, \ldots, n$. makes it a metabelian Lie algebra. Due to the universal property of the free metabelian Lie algebra F_n in the variety $\mathfrak{A}^2$, we may extend every mapping $X_n \to (KU_n)\mathrm{wr}(KT_n)$ to a homomorphism $F_n \to (KU_n)\mathrm{wr}(KT_n)$. Now the homomorphism $\delta : F_n \to (KU_n)\mathrm{wr}(KT_n)$ defined by

$$\delta : x_i \to u_i + t_i, \quad i = 1, \ldots, n,$$

is a monomorphism as a special case of the embedding theorem of Shmel'kin [14]. Consequently, if

$$f = \sum_{i>j} [x_i, x_j] p_{ij}(X_n) \in F_n', \quad p_{ij}(X_n) \in K[X_n],$$

then

$$\delta(f) = \sum_{i>j} (u_i x_j - u_j x_i) p_{ij}(X_n) \in W_n.$$

Furthermore, $\delta^{-1}(w) \in F_n'$ where w is of the form (1) if and only if $w \in W_n$, and by [14, Theorem 2] we have the following.

$$\sum_{i=1}^{n} x_i p_i(X_n) = 0. \tag{2}$$

2.2 Symmetric polynomials

Recall that a polynomial $p(X_n) \in F_n$ is symmetric if

$$p(x_1, \ldots, x_n) = \pi \cdot p(x_1, \ldots, x_n) = p(x_{\pi(1)}, \ldots, x_{\pi(n)})$$

for all $\pi \in S_n$. Due to the fact that $\delta((F_n')^{S_n}) \in W_n^{S_n}$, it sufficies to work in the free module W_n. We assume that S_n acts on W_n as follows.

$$\pi \cdot w = \pi \cdot \sum_{i=1}^{n} u_i\, p_i(X_n) = \sum_{i=1}^{n} u_{\pi(i)}\, p_i(x_{\pi(1)}, \ldots, x_{\pi(n)})$$

In the sequel, we fix some notations given below.

$$e_q = \sum x_{j_1} \cdots x_{j_q}, \quad 1 \le j_1 < \cdots < j_q \le n,\ q = 1, \ldots, n; \tag{3}$$

$$e_q^{(i)} = \sum_{j_l \neq i} x_{j_1} \cdots x_{j_q}, \quad 1 \le j_1 < \cdots < j_q \le n,\ q = 1, \ldots, n-1; \tag{4}$$

$$e_q^{(ij)} = \sum_{j_l \notin \{i,j\}} x_{j_1} \cdots x_{j_q}, \quad 1 \le j_1 < \cdots < j_q \le n,\ q = 1, \ldots, n-2; \tag{5}$$

$$\varepsilon_{1+q} = \sum_{i=1}^{n} \sum_{j_1 < \cdots < j_q} u_i x_{j_1} \cdots x_{j_q}, \quad i \neq j_1, \ldots, j_q,\ q = 0, 1, \ldots, n-1.$$

$$e_k = 0 \ \text{ for } \ k < 0\,; \ e_k^{(i)} = 0 \ \text{ for } \ k < 0 \text{ or } k \ge n\,;$$

$$e_k^{(ij)} = 0 \ \text{ for } \ k < 0 \text{ or } k \ge n-1\,; \ e_0 = 1\,; \ \ e_0^{(i)} = 1\,; \ \ e_0^{(ij)} = 1.$$

The following two results are due to [6], where the corollary gives a set of generators of $K[X_n]^{S_n}$-module $W_n^{S_n}$ while an implicit expression of a generating set of the $K[X_n]^{S_n}$-module $(F_n')^{S_n}$ was provided in the theorem.

Corollary 1 *The $K[X_n]^{S_n}$-module $W_n^{S_n}$ is generated by the polynomials ε_{1+q}, $q = 0, 1, \ldots, n-1$.*

Theorem 2 *The $K[X_n]^{S_n}$-module $(F_n')^{S_n}$ is generated by the polynomials $g_{ij} = \delta^{-1}(h_{ij})$, $1 \le i < j \le n$, where*

$$h_{ij} = j\varepsilon_i e_j - i\varepsilon_j e_i. \tag{6}$$

3 Main results

3.1 Explicit generators

Our first result exhibits the explicit generators of the $K[X_n]^{S_n}$-module $(F_n')^{S_n}$.

Theorem 3 *The $K[X_n]^{S_n}$-module $(F_n')^{S_n}$ is generated by*

$$g_{ij} = \sum_{1 \le k < l \le n} [x_l, x_k](x_l - x_k)(e_{i-1}^{(kl)} e_{j-2}^{(kl)} - e_{j-1}^{(kl)} e_{i-2}^{(kl)})$$

where $1 \le i < j \le n$.

Proof Initially, we have by Equations (3), (4), and (5) that

$$e_q = e_q^{(i)} + x_i e_{q-1}^{(i)} \; ; \;\; e_q^{(i)} = e_q^{(ij)} + x_j e_{q-1}^{(ij)} \, . \tag{7}$$

We also obtain the followings by direct computations.

$$je_j = x_1 e_{j-1}^{(1)} + \cdots + x_n e_{j-1}^{(n)} \, , \;\; \forall j \in \{1, \ldots, n\}. \tag{8}$$

We have to rearrange the expression of h_{ij} in order to obtain its inverse image and confirm the equality $\delta^{-1}(h_{ij}) = g_{ij}$. Now let

$$h_{ij} = u_1 h_{ij1} + \cdots + u_n h_{ijn} \tag{9}$$

where $h_{ijk} \in K[X_n]$ are the coefficients of free generators u_k of the free $K[X_n]$-module W_n. Computations give by Theorem 2, Equations (7) and (8) that

$$\begin{aligned} h_{ij1} =& (je_j\varepsilon_i - ie_i\varepsilon_j)_{u_1} = je_j e_{i-1}^{(1)} - ie_i e_{j-1}^{(1)} \\ =& (x_1 e_{j-1}^{(1)} + \cdots + x_n e_{j-1}^{(n)}) e_{i-1}^{(1)} - (x_1 e_{i-1}^{(1)} + \cdots + x_n e_{i-1}^{(n)}) e_{j-1}^{(1)} \\ =& (x_2 e_{j-1}^{(2)} + \cdots + x_n e_{j-1}^{(n)}) e_{i-1}^{(1)} - (x_2 e_{i-1}^{(2)} + \cdots + x_n e_{i-1}^{(n)}) e_{j-1}^{(1)} \\ =& A_{ij} - A_{ji} \end{aligned}$$

where

$$\begin{aligned} A_{ij} =& (x_2 e_{j-1}^{(2)} + \cdots + x_n e_{j-1}^{(n)}) e_{i-1}^{(1)} \\ =& x_2 (e_{j-1}^{(12)} + x_1 e_{j-2}^{(12)})(e_{i-1}^{(12)} + x_2 e_{i-2}^{(12)}) + x_3 (e_{j-1}^{(13)} + x_1 e_{j-2}^{(13)})(e_{i-1}^{(13)} + x_3 e_{i-2}^{(13)}) \\ & + \cdots + x_n (e_{j-1}^{(1n)} + x_1 e_{j-2}^{(1n)})(e_{i-1}^{(1n)} + x_n e_{i-2}^{(1n)}) \end{aligned}$$

since $e_{i-1}^{(1)} = e_{i-1}^{(12)} + x_2 e_{i-2}^{(12)} = \cdots = e_{i-1}^{(1n)} + x_n e_{i-2}^{(1n)}$. Thus, we have that

$$\begin{aligned} h_{ij1} =& - x_2(x_2 - x_1)(e_{i-1}^{(12)} e_{j-2}^{(12)} - e_{j-1}^{(12)} e_{i-2}^{(12)}) - x_3(x_3 - x_1)(e_{i-1}^{(13)} e_{j-2}^{(13)} - e_{j-1}^{(13)} e_{i-2}^{(13)}) \\ & - \cdots - x_n(x_n - x_1)(e_{i-1}^{(1n)} e_{j-2}^{(1n)} - e_{j-1}^{(1n)} e_{i-2}^{(1n)}) \end{aligned}$$

and in general for any fixed $k \in \{1, \ldots, n\}$, we obtain that

$$h_{ijk} = \sum_{l}^{l \neq k} -x_l(x_l - x_k)(e_{i-1}^{(kl)} e_{j-2}^{(kl)} - e_{j-1}^{(kl)} e_{i-2}^{(kl)}). \tag{10}$$

Finally,

$$h_{ij} = \sum_{k=1}^{n} u_k h_{ijk} = \sum_{1 \leq k < l \leq n} (u_l x_k - u_k x_l)(x_l - x_k)(e_{i-1}^{(kl)} e_{j-2}^{(kl)} - e_{j-1}^{(kl)} e_{i-2}^{(kl)}) \tag{11}$$

which completes the proof. □

Example 1 Let us illustrate the generators of g_{ij} for $n = 2, 3, 4$.
$* n = 2$. $K[X_2]^{S_2} = K[e_1, e_2]$, and

$$g_{12} = [x_2, x_1](x_2 - x_1).$$

$* n = 3$. $K[X_3]^{S_3} = K[e_1, e_2, e_3]$, and

$$\begin{aligned}
g_{12} &= [x_2, x_1](x_2 - x_1) + [x_3, x_1](x_3 - x_1) + [x_3, x_2](x_3 - x_2)\\
g_{13} &= [x_2, x_1](x_2 - x_1)x_3 + [x_3, x_1](x_3 - x_1)x_2 + [x_3, x_2](x_3 - x_2)x_1\\
g_{23} &= [x_2, x_1](x_2 - x_1)x_3^2 + [x_3, x_1](x_3 - x_1)x_2^2 + [x_3, x_2](x_3 - x_2)x_1^2
\end{aligned}$$

$* n = 4$. $K[X_3]^{S_3} = K[e_1, e_2, e_3, e_4]$, and

$$\begin{aligned}
g_{12} = &[x_2, x_1](x_2 - x_1) + [x_3, x_1](x_3 - x_1) + [x_3, x_2](x_3 - x_2)\\
&+ [x_4, x_1](x_4 - x_1) + [x_4, x_2](x_4 - x_2) + [x_4, x_3](x_4 - x_3)
\end{aligned}$$

$$\begin{aligned}
g_{13} = &[x_2, x_1](x_2 - x_1)(x_3 + x_4) + [x_3, x_1](x_3 - x_1)(x_2 + x_4)\\
&+ [x_3, x_2](x_3 - x_2)(x_1 + x_4) + [x_4, x_1](x_4 - x_1)(x_2 + x_3)\\
&+ [x_4, x_2](x_4 - x_2)(x_1 + x_3) + [x_4, x_3](x_4 - x_3)(x_1 + x_2)
\end{aligned}$$

$$\begin{aligned}
g_{14} = &[x_2, x_1](x_2 - x_1)x_3x_4 + [x_3, x_1](x_3 - x_1)x_2x_4\\
&+ [x_3, x_2](x_3 - x_2)x_1x_4 + [x_4, x_1](x_4 - x_1)x_2x_3\\
&+ [x_4, x_2](x_4 - x_2)x_1x_3 + [x_4, x_3](x_4 - x_3)x_1x_2
\end{aligned}$$

$$\begin{aligned}
g_{23} = &[x_2, x_1](x_2 - x_1)\left((x_3 + x_4)^2 - x_3x_4\right)\\
&+ [x_3, x_1](x_3 - x_1)\left((x_2 + x_4)^2 - x_2x_4\right)\\
&+ [x_3, x_2](x_3 - x_2)\left((x_1 + x_4)^2 - x_1x_4\right)\\
&+ [x_4, x_1](x_4 - x_1)\left((x_2 + x_3)^2 - x_2x_3\right)\\
&+ [x_4, x_2](x_4 - x_2)\left((x_1 + x_3)^2 - x_1x_3\right)\\
&+ [x_4, x_3](x_4 - x_3)\left((x_1 + x_2)^2 - x_1x_2\right)
\end{aligned}$$

$$\begin{aligned}
g_{24} = &[x_2, x_1](x_2 - x_1)(x_3 + x_4)x_3x_4 + [x_3, x_1](x_3 - x_1)(x_2 + x_4)x_2x_4\\
&+ [x_3, x_2](x_3 - x_2)(x_1 + x_4)x_1x_4 + [x_4, x_1](x_4 - x_1)(x_2 + x_3)x_2x_3\\
&+ [x_4, x_2](x_4 - x_2)(x_1 + x_3)x_1x_3 + [x_4, x_3](x_4 - x_3)(x_1 + x_2)x_1x_2
\end{aligned}$$

$$\begin{aligned} g_{34} = &[x_2, x_1](x_2 - x_1)(x_3x_4)^2 + [x_3, x_1](x_3 - x_1)(x_2x_4)^2 \\ &+ [x_3, x_2](x_3 - x_2)(x_1x_4)^2 + [x_4, x_1](x_4 - x_1)(x_2x_3)^2 \\ &+ [x_4, x_2](x_4 - x_2)(x_1x_3)^2 + [x_4, x_3](x_4 - x_3)(x_1x_2)^2 \end{aligned}$$

3.2 A presentation

Note that in Remark 10 of [6], a presentation of the $K[X_n]^{S_n}$-module $(F_n')^{S_n}$ was asked, and the relation

$$kg_{ij}e_k - jg_{ik}e_j + ig_{jk}e_i = 0, \tag{12}$$

were provided for $1 \leq i < j < k \leq n$. In this section, we were motivated by this question and gave a presentation of the $K[X_n]^{S_n}$-module $(F_n')^{S_n}$ with respect to the generators g_{ij}. Furthermore, we computed the Hilbert series $H((F_n')^{S_n}, t)$ of the K-space $(F_n')^{S_n}$.

Let $E_n = \{e_1, \ldots, e_n\}$ and $E_n^{ij} = E_n \setminus \{e_{i+1}, \ldots, e_{j-1}\}$, where $i + 1 < j$. Then we have the following result.

Theorem 4 *Every symmetric polynomial $p \in (F_n')^{S_n}$ can be expressed as*

$$p = \sum_{i=1}^{n-1} g_{i(i+1)}r_{i(i+1)} + \sum_{i+1<j} g_{ij}s_{ij} \tag{13}$$

for uniquely determined $r_{i(i+1)} \in K[E_n]$, $s_{ij} \in K[E_n^{ij}]$.

Proof We perform the proof in two steps: The expression of Equation (13), and the uniqueness.

Step 1. We know by [6] that the $K[X_n]^{S_n}$-module $(F')^{S_n}$ is generated by g_{ij}, $1 \leq i < j \leq n$. On the other hand, by Equation (12) we have that e_jg_{ik} is contained in the $K[X_n]^{S_n}$-module generated by g_{ij} and g_{jk} for $1 \leq i < j < k \leq n$; that is

$$e_jg_{ik} \in \langle g_{ij}, g_{jk}\rangle_{K[X_n]^{S_n}} \tag{14}$$

Now let $p \in (F_n')^{S_n}$ be an arbitrary symmetric polynomial. Then p is of the form

$$p = \sum_{1\leq i<j\leq n} g_{ij}v_{ij}$$

for some $v_{ij} \in K[E_n]$ by [6]. We have to show that p can be expressed as in Equation (13). If $n = 2$, then there is nothing to prove. Let $n \geq 3$. We have that $v_{1n} = v_{1n}^{(2)} + e_2w_{1n}^{(2)}$ for some $v_{1n}^{(2)} \in K[E_n^{13}] = K[E_n \setminus \{e_2\}]$, $w_{1n}^{(2)} \in K[E_n]$. Therefore, $g_{1n}v_{1n} = g_{1n}(v_{1n}^{(2)} + e_2w_{1n}^{(2)})$, and thus

$$g_{1n}v_{1n} = g_{1n}v_{1n}^{(2)} \mod\langle g_{12}, g_{2n}\rangle$$

by Equation (14). Similarly, $v_{1n}^{(2)} = v_{1n}^{(3)} + e_3 w_{1n}^{(3)}$ for some $v_{1n}^{(3)} \in K[E_n^{14}]$, $w_{1n}^{(3)} \in K[E_n]$. Hence,

$$g_{1n}v_{1n} = g_{1n}v_{1n}^{(3)} \mod\langle g_{13}, g_{3n}\rangle.$$

This process implies

$$g_{1n}v_{1n} = g_{1n}v_{1n}^{(n-1)} \mod\langle g_{1(n-1)}, g_{(n-1)n}\rangle.$$

for some $v_{1n}^{(n-1)} \in K[E_n^{1n}]$. One may assume by similar steps that the coefficient v_{jn} of g_{jn} is contained in $K[E_n^{jn}]$; and in general $v_{jk} \in K[E_n^{jk}]$ for $j+1 < k$.

Step 2. Initially, recall by Equation (6) that

$$\delta(g_{ij}) = h_{ij} = je_j\varepsilon_i - ie_i\varepsilon_j\ ,\ \ 1 \le i < j \le n.$$

Now let

$$0 = \sum_{i=1}^{n-1} g_{i(i+1)}r_{i(i+1)} + \sum_{k+1<l} g_{kl}s_{kl}$$

for some $r_{i(i+1)} \in K[E_n]$, $s_{kl} \in K[E_n^{kl}]$. We have to show that

$$r_{i(i+1)} = s_{kl} = 0\ ,\ \ i = 1, \ldots, n-1\ ,\ \ k+1 < l.$$

Note: We need to keep in mind that

$$\begin{aligned} e_q^{(1)} - (1j)e_q^{(1)} &= e_q^{(1)} - e_q^{(j)} \\ &= (x_j e_{q-1}^{(1j)} + e_q^{(1j)}) - (x_1 e_{q-1}^{(1j)} + e_q^{(1j)}) \\ &= (x_j - x_1)e_{q-1}^{(1j)}\ ,\ \ 2 \le j \le n,\ 1 \le q \le n, \end{aligned}$$

for the transposition $(1j) \in S_n$. This can be generalized as follows.

$$e_q^{(1,\ldots,k)} - (1(k+1))e_q^{(1,\ldots,k)} = (x_{k+1} - x_1)e_{q-1}^{(1,\ldots,k,k+1)}$$

for $1 \le k \le n-1$.

In the sequel, we initially perform computations for $n = 2, 3, 4$ as an illustration of an algorithm solving the equation.

$*$ Assume that $n = 2$ and $g_{12}r_{12} = 0$, for some $r_{12} \in K[E_2] = K[e_1, e_2]$, which implies that $\delta(g_{12}r_{12}) = h_{12}r_{12} = 0$ and

$$0 = (2e_2\varepsilon_1 - e_1\varepsilon_2)r_{12} = (2e_2(u_1+u_2) - e_1(u_1e_1^{(1)} + u_2e_2^{(2)}))r_{12}$$

in the free module W_2 generated by u_1 and u_2. That is,

$$h_{121}r_{12} = (2e_2 - e_1^{(1)}e_1)r_{12} = 0$$

where h_{121} is the well defined coefficient of u_1 in the expression of h_{12}. Thus

$$p_1 = h_{121}r_{12} - 2e_2r_{12} = -e_1^{(1)}e_1r_{12} \in K[E_2]$$

is a symmetric polynomial, and it is preserved under the action of the transposition (12). Therefore,

$$0 = p_1 - (12)p_1 = -(e_1^{(1)} - e_1^{(2)})e_1r_{12} = -(x_2 - x_1)e_1r_{12}$$

Hence, $r_{12} = 0$ in the integral domain $K[X_2]$, since $(x_2 - x_1)e_1 \neq 0$.
$*$ Let $n = 3$ and

$$g_{12}r_{12} + g_{13}s_{13} + g_{23}r_{23} = 0$$

where $r_{12}, r_{23} \in K[E_3]$, and $s_{13} \in K[E_3^{13}] = K[e_1, e_3]$. Similarly we have

$$h_{121}r_{12} + h_{131}s_{13} + h_{231}r_{23} = 0,$$

that is

$$(2e_2 - e_1^{(1)}e_1)r_{12} + (3e_3 - e_2^{(1)}e_1)s_{13} + (3e_1^{(1)}e_3 - 2e_2^{(1)}e_2)r_{23} = 0.$$

Then, by the fact that

$$p_1 = -e_1^{(1)}e_1r_{12} - e_2^{(1)}e_1s_{13} + (3e_1^{(1)}e_3 - 2e_2^{(1)}e_2)r_{23} \in K[E_3]$$

we have

$$0 = p_1 - (12)p_1 \overset{x_2-x_1}{=} -e_1r_{12} - e_1^{(12)}e_1s_{13} + (3e_3 - 2e_1^{(12)}e_2)r_{23} \tag{15}$$

and get

$$p_2 = -e_1^{(12)}e_1s_{13} - 2e_1^{(12)}e_2r_{23}$$

is symmetric. However,

$$0 = p_2 - (13)p_2 \overset{x_3-x_1}{=} -e_1s_{13} - 2e_2r_{23}.$$

This implies e_2 divides $s_{13} \in K[e_1, e_3]$; that is, $s_{13} = 0$ and hence $r_{23} = 0$. Now going back to Equation (15) we have $r_{12} = 0$.
$*$ Let $n = 4$ and

$$g_{12}r_{12} + g_{13}s_{13} + g_{23}r_{23} + g_{14}s_{14} + g_{24}s_{24} + g_{34}r_{34} = 0$$

where $r_{12}, r_{23}, r_{34} \in K[E_4]$, and $s_{ij} \in K[E_4^{ij}]$. We obtain the followings, by similar steps.

$$0 = (2e_2 - e_1^{(1)}e_1)r_{12} + (3e_3 - e_2^{(1)}e_1)s_{13} + (3e_1^{(1)}e_3 - 2e_2^{(1)}e_2)r_{23}$$

$$+ (4e_4 - e_3^{(1)}e_1)s_{14} + (4e_1^{(1)}e_4 - 2e_3^{(1)}e_2)s_{24} + (4e_2^{(1)}e_4 - 3e_3^{(1)}e_3)r_{34}$$

$$\begin{aligned} p_1 = &-e_1^{(1)}e_1r_{12} + (3e_1^{(1)}e_3 - 2e_2^{(1)}e_2)r_{23} - e_2^{(1)}e_1s_{13} \\ &- e_3^{(1)}e_1s_{14} + (4e_1^{(1)}e_4 - 2e_3^{(1)}e_2)s_{24} + (4e_2^{(1)}e_4 - 3e_3^{(1)}e_3)r_{34} \in K[E_4] \end{aligned}$$

$$\begin{aligned} 0 = p_1 - (12)p_1 \overset{x_2-x_1}{=} &-e_1r_{12} + (3e_3 - 2e_1^{(12)}e_2)r_{23} - e_1^{(12)}e_1s_{13} \\ &- e_2^{(12)}e_1s_{14} + (4e_4 - 2e_2^{(12)}e_2)s_{24} + (4e_1^{(12)}e_4 - 3e_2^{(12)}e_3)r_{34} \end{aligned}$$

$$\begin{aligned} p_2 = &- 2e_1^{(12)}e_2r_{23} - e_1^{(12)}e_1s_{13} \\ &- e_2^{(12)}e_1s_{14} - 2e_2^{(12)}e_2s_{24} + (4e_1^{(12)}e_4 - 3e_2^{(12)}e_3)r_{34} \in K[E_4] \end{aligned}$$

$$\begin{aligned} 0 = p_2 - (13)p_2 \overset{x_3-x_1}{=} &-2e_2r_{23} - e_1s_{13} \\ &- e_1^{(123)}e_1s_{14} - 2e_1^{(123)}e_2s_{24} + (4e_4 - 3e_1^{(123)}e_3)r_{34} \end{aligned}$$

$$p_3 = - e_1^{(123)}e_1s_{14} - 2e_1^{(123)}e_2s_{24} - 3e_1^{(123)}e_3r_{34} \in K[E_4]$$

and finally

$$0 = p_3 - (14)p_3 \overset{x_4-x_1}{=} -e_1s_{14} - 2e_2s_{24} - 3e_3r_{34}$$

That is,

$$e_1s_{14} + 2e_2s_{24} + 3e_3r_{34} = 0 \tag{16}$$

where $s_{14} = s_{14}(e_1, e_4)$, $s_{24} = s_{24}(e_1, e_2, e_4)$, $r_{34} = r_{34}(e_1, e_2, e_3, e_4)$. Equation 16 implies that $e_3 \mid e_1s_{14}+2e_2s_{24}$ where $e_1s_{14}+2e_2s_{24}$ does not depend on e_3 in the algebra $K[e_1, e_2, e_3, e_4]$ generated by algebraically independent variables e_1, e_2, e_3, e_4. That is, $e_1s_{14} + 2e_2s_{24} = 0$, and that $e_2 \mid e_1s_{14}$ which does not depend on e_2. Therefore, $s_{14} = 0$, and inductively $s_{24} = r_{34} = 0$. Now going back to the previous step, the process works as in the case $n = 3$: Firstly $s_{13} = r_{23} = 0$, and later $r_{12} = 0$.
$*$ In general, let $n \geq 2$ and

$$0 = \sum_{i=1}^{n-1} g_{i(i+1)}r_{i(i+1)} + \sum_{k+1<l} g_{kl}s_{kl}$$

for some $r_{i(i+1)} \in K[E_n]$, $s_{kl} \in K[E_n^{kl}]$. Then the algorithm defined above implies, after $n - 1$ steps that

$$\sum_{i=1}^{n-2} ie_is_{in} = (1 - n)e_{n-1}r_{(n-1)n}$$

where the expression on the left does not depend on e_{n-1}. Thus

$$\sum_{i=1}^{n-3} i e_i s_{in} = (2-n) e_{n-2} s_{(n-1)n}$$

where the expression on the left does not depend on e_{n-2}. Following this process, one obtains $s_{1n} = 0, \ldots, s_{(n-1)n} = 0$, respectively. Later, $r_{(n-1)n} = 0$ is a direct consequence. Inductively, one shows

$$s_{1j} = 0, \ldots, s_{(j-2)j} = 0, r_{(j-1)j} = 0$$

where $j = n-1, \ldots, j = 2$, respectively. □

Similar steps used in the proof of Theorem 4 results in the corollary below.

Corollary 5 *The generators $g_{i(i+1)}$, $1 \leq i \leq n-1$, are free in the $K[X_n]^{S_n}$-module $(F')^{S_n}$; i.e., the submodule generated by $g_{12}, \ldots, g_{(n-1)n}$ is a free right $K[X_n]^{S_n}$-module of rank $n-1$.*

3.3 The Hilbert series

It is well known that the free metabelian Lie algebra F_n is a graded algebra and it can be expressed as

$$F_n = \bigoplus_{k \geq 1} F_n^{(k)}$$

where $F_n^{(k)}$ stands for the homogeneous subspace of degree k of F_n. Similarly, $(F_n')^{S_n}$ is graded, and

$$(F_n')^{S_n} = \bigoplus_{k \geq 3} ((F_n')^{S_n})^{(k)}$$

since $((F_n')^{S_n})^{(1)} = ((F_n')^{S_n})^{(2)} = 0$ as a result of Theorem 4. Therefore, the Hilbert series of $(F_n')^{S_n}$ can be formalized as

$$H((F_n')^{S_n}, t) \sum_{k \geq 3} \dim\left(((F_n')^{S_n})^{(k)}\right) t^k$$

The next theorem is our last result of the paper.

Theorem 6 *The Hilbert series $H((F_n')^{S_n}, t)$ of the K-space $(F_n')^{S_n}$ is*

$$H((F_n')^{S_n}, t) = \left(\sum_{i=1}^{n-1} t^{2i+1} + \sum_{i=0}^{n-3} \sum_{j=0}^{n-i-3} t^{2i+j+4} \prod_{k=0}^{j} (1 - t^{k+i+2}) \right) \prod_{i=1}^{n} \frac{1}{1-t^i}$$

Proof Theorem 4 can be expressed as follows: The $K[E_n]$-module $(F_n')^{S_n}$ has the presentation

$$(F_n')^{S_n} = \bigoplus_{i=1}^{n-1} g_{i(i+1)} K[E_n] \oplus \bigoplus_{i+1<j} g_{ij} K[E_n^{ij}]$$

Therefore,

$$\begin{aligned} H((F_n')^{S_n}, t) &= \sum_{i=1}^{n-1} t^{\deg(g_{i(i+1)})} H(K[E_n], t) + \sum_{\substack{1\le i\le n-2 \\ i+2\le j\le n}} t^{\deg(g_{ij})} H(K[E_n^{ij}], t) \\ &= \sum_{i=1}^{n-1} t^{2i+1} \prod_{i=1}^{n} \frac{1}{1-t^i} + \sum_{\substack{1\le i\le n-2 \\ i+2\le j\le n}} t^{i+j} \prod_{\substack{1\le k\le i \\ j\le k\le n}} \frac{1}{1-t^k} \\ &= \left(\sum_{i=1}^{n-1} t^{2i+1} + H_n \right) \prod_{i=1}^{n} \frac{1}{1-t^i} \end{aligned}$$

where

$$\begin{aligned} H_n &= \sum_{i=1}^{n-2} \sum_{j=i+2}^{n} t^{i+j} \prod_{k=i+1}^{j-1} (1-t^k) = \sum_{i=0}^{n-3} \sum_{j=i+3}^{n} t^{i+j+1} \prod_{k=i+2}^{j-1} (1-t^k) \\ &= \sum_{i=0}^{n-3} \sum_{j=0}^{n-i-3} t^{2i+j+4} \prod_{k=i+2}^{j+i+2} (1-t^k) = \sum_{i=0}^{n-3} \sum_{j=0}^{n-i-3} t^{2i+j+4} \prod_{k=0}^{j} (1-t^{k+i+2}) \end{aligned}$$

and we are done. □

Author Contributions E.A., Ş.F., and N.Ş.Ö. wrote the main manuscript text.

Funding Open access funding provided by the Scientific and Technological Research Council of Türkiye (TÜBİTAK). None.

Data Availability Data sharing not applicable to this article as no data sets were generated or analyzed during the current study.

Declarations

Ethical Statement/Conflict of Interest The study is in compliance with Ethical Standards. On behalf of all authors, the corresponding author states that there is no conflict of interest.

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