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Research Article

A Novel Application of Expansive Mappings and Cone Theory for Third-Order Impulsive Dynamic Equations

Svetlin G. Georgiev[1], Sibel Doğru Akgöl[2,*]

[1]Department of Mathematics, Sorbonne University, Paris, France
[2]Department of Mathematics, Faculty of Arts and Sciences, Çukurova University, Adana, Türkiye

ABSTRACT

We establish novel existence criteria for solutions to a class of third-order impulsive nonlinear dynamic equations on time scales, subject to homogeneous boundary conditions. Although first and second-order impulsive dynamic equations are well studied, the analysis of third-order problems remains comparatively underdeveloped. To address this gap, we employ very recent advancements in fixed point theory, utilizing innovative concepts involving expansive mappings and operator addition. This approach enables us to first prove the existence of at least one solution and subsequently demonstrate the presence of at least two distinct solutions. A significant advantage of our methodology is that it does not impose restrictions on boundedness on the nonlinear components of the equation, a requirement common in most existing studies. The applicability and practical relevance of our theoretical results are validated through an illustrative example.

Keywords: BVPs, dynamic equation, impulse effects, expansive map, time scale
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***Corresponding Author:** Sibel Doğru Akgöl
svetlingeorgiev1@gmail.com (S. G. Georgiev) 0000-0001-8015-4226 https://ror.org/02en5vm52
sakgol@cu.edu.tr (S. D. Akgöl) 0000-0003-3513-1046 https://ror.org/05wxkj555

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1 Introduction

A cornerstone in the unification of continuous and discrete analysis is the theory of dynamic equations on time scales, pioneered by Stefan Hilger in his seminal 1988 work. This powerful framework provides a robust methodology for modeling phenomena that inherently blend continuous and discrete dynamics, finding profound applications across biology, physics, and engineering. By offering a unified formalism that seamlessly generalizes standard differential and difference equations, the time scales calculus has facilitated significant advances in the analysis of complex dynamic systems.

Concurrently, the theory of impulsive differential equations has garnered considerable attention for its capacity to model systems experiencing sudden, discontinuous changes at isolated points in time. Such models are indispensable in fields like epidemiology, neural networks, and control theory, where external interventions or triggering events cause abrupt state transitions. The integration of impulsive effects with

the time scales calculus has further enriched this modeling paradigm, enabling a more nuanced description of systems that evolve both continuously and through instantaneous jumps.

Despite these developments, the study of higher-order impulsive dynamic equations on time scales remains relatively nascent. While first and second-order problems have been extensively explored, the qualitative theory of third-order equations with impulse effects is markedly less developed. In particular, establishing existence criteria for solutions to third-order boundary value problems (BVPs) on time scales with nonlinearities and impulses presents a challenging and sparsely populated area of research.

A survey of the literature reveals numerous contributions to third-order dynamic equations on time scales, encompassing neutral equations [20, 21], problems involving the p-Laplacian [13], classical dynamic equations [23, 25, 26], and even partial dynamic equations [10]. In contrast, the body of work dedicated to their impulsive counterparts is exceedingly limited. Among the few relevant studies, the authors of [14, 15] established the existence of positive solutions for third-order impulsive BVPs with p-Laplacian. Meanwhile, [11, 18] employed Schauder's fixed point theorem to prove existence for certain classes of impulsive problems, with [11] addressing a multi-point BVP. Some recent work on the existence of solutions and the general theory of dynamic equations on time scales can be found in the edited book [2], and in papers [12, 22, 24].

The conspicuous gap in the literature regarding higher-order impulsive dynamics, coupled with the broad applicability of such models, motivates the present work. Previous studies on multi-point boundary value problems such as the third-order impulsive dynamic equation investigated in [11] and the periodic boundary conditions examined by Li and Li [18] have primarily established existence results through the classical Schauder fixed point theorem. Although effective, these two papers impose significantly more restrictive conditions on the nonlinear functions. In this paper, we advance the existing research in two significant directions. First, we substantially relax the hypotheses on the right-hand side functions, thereby embedding earlier frameworks as particular instances of our more general setting. Second, and more fundamentally, we utilize more recent and sophisticated mathematical tools instead of conventional fixed-point methodologies. Specifically, we employ the theory of expansive mappings on cones, which is a more modern approach that allows us to handle weaker conditions and provides new insights into the structure of solutions. By leveraging this novel approach under milder conditions, we derive new existence results that are both more general and widely applicable than those currently available. Our findings not only extend existing theory, but also offer a fresh perspective and a more versatile tool for analyzing complex systems governed by higher-order impulsive dynamics.

Before presenting our main problem, we list the essential definitions and concepts of time scale calculus. The theory of time scales begins by defining $\mathbb{T}$ as an arbitrary nonempty closed subset of $\mathbb{R}$. Central to the calculus on $\mathbb{T}$ are the forward and backward jump operators, defined by $\sigma(\xi) = \inf\{s \in \mathbb{T} : s > \xi\}$ and $\rho(\xi) = \sup\{s \in \mathbb{T} : s < \xi\}$, respectively. These are used to define the composition $f^{\sigma} = f \circ \sigma$.

A point $\xi \in \mathbb{T}$ is considered as

- right-dense when $\sigma(\xi) = \xi$ and ξ is not the maximum element of $\mathbb{T}$;
- left-dense when $\rho(\xi) = \xi$ and ξ is not the minimum element of $\mathbb{T}$.

Note that condition $\xi \neq \max \mathbb{T}$ is required for ξ to be a right-dense point. This ensures that the Δ-derivative is well-defined, as a maximum element lacks a right-side neighborhood within the time scale for the limiting process. Similarly, the condition $\xi \neq \min \mathbb{T}$ for left-density ensures the existence of a left neighborhood, allowing for a consistent definition of backward operators and limits.

Moreover, a function f is called rd-continuous if it is continuous at every right-dense point and possesses finite left-sided limits at all left-dense points. The derivative of f on a time scale is symbolized by f^{Δ}. For

a complete treatment of these concepts, we refer to the standard texts [1, 4, 5].

To achieve our goal, we will investigate the following BVP:

$$\begin{cases} \eta^{\Delta^3}(\xi) + f(\xi, \eta(\xi)) &= 0, \quad \xi \in J_0, \\ \eta(\xi_j^+) - \eta(\xi_j^-) &= I_j(\eta(\xi_j)), \\ \eta^{\Delta}(\xi_j^+) - \eta^{\Delta}(\xi_j^-) &= L_j(\eta(\xi_j)), \quad j \in \{1, \ldots, m\}, \\ \eta(0) \;=\; \eta(\xi_*) \;=\; \eta^{\Delta^2}(0) &= 0, \end{cases} \tag{1}$$

where $\zeta(\tau\pm) = \lim_{t\to\tau\pm} \zeta(t)$; $\mathbb{T}$ is a time scale with the forward jump operator σ and the delta differentiation operator Δ; $0, \xi_* \in \mathbb{T}$; $\xi_* > 0$; $J = [0, \xi_*]_{\mathbb{T}} := \mathbb{T} \cap [0, \xi_*]$ and

$$0 = \xi_0 < \xi_1 < \ldots < \xi_m < \xi_{m+1} = \xi_*$$

such that the impulse moments ξ_j, $j \in \{1, \ldots m\}$ are right-dense, $J_0 = J \backslash \{\xi_j\}_{j=1}^m$, and

(B1) $f \in C([0, \xi_*]_{\mathbb{T}} \times \mathbb{R})$ satisfies the condition

$$|f(\xi, \eta)| \le p(\xi) + q(\xi)|\eta|^l, \quad (\xi, \eta) \in [0, \xi_*]_{\mathbb{T}} \times \mathbb{R}$$

for some non-negative functions $p, q \in C([0, \xi_*]_{\mathbb{T}})$ for which $p, q \le B$ on $[0, \xi_*]_{\mathbb{T}}$, and for some $l \ge 0$,

(B2) $I_j, L_j \in C(\mathbb{R})$,

$$|I_j(\eta)| \le c_j|\eta|^{\gamma_j}, \quad |L_j(\eta)| \le d_j|\eta|^{\beta_j}, \quad j \in \{1, \ldots, m\}, \quad \eta \in \mathbb{R},$$

where $c_j, d_j, \gamma_j, \beta_j$ are nonnegative constants, $j \in \{1, \ldots, m\}$.

This manuscript is structured in the following manner: The next section introduces the preliminaries that involve the necessary tools of time scale calculus, recent fixed point theorems, and an integral representation of the BVP (1). In Sections 3 and 4, we state and prove our main results regarding the existence of at least one and at least two solutions, respectively. In Section 5, we give an example to illustrate our main results.

2 Preliminary Results

We begin by establishing our setting: Let $\mathcal{B}$ denote a Banach space over the real numbers. To facilitate the subsequent discussion, we first revisit several fundamental concepts from nonlinear functional analysis.

Definition 1 ([7]). *A mapping $\mathcal{T} : S \subset \mathcal{B} \to \mathcal{B}$ is termed compact provided that its image, $\mathcal{T}(S)$, is a subset of some compact set in $\mathcal{B}$. If $\mathcal{T}$ is both continuous and transforms every bounded subset of its domain into a set with compact closure, then it is called completely continuous.*

Our main existence argument will be grounded on a fixed point principle. For a proof of the following result, the reader is referred to [8, 16].

Theorem 1. *Let E be a subset of a Banach space $\mathcal{B}$ such that E is closed and convex. Let also S be an open subset of E containing the origin. Assume the operators $\mathcal{T}$ and $\mathcal{F}$ satisfy:*

- $\mathcal{T}(\xi) = \varepsilon\xi$ *for all* $\xi \in \overline{S}$ *and some* $\varepsilon > 1$,
- *The mapping* $\mathcal{I} - \mathcal{F} : \overline{S} \to \mathcal{B}$ *is continuous, compact, and*
- *The equation* $\xi = \lambda(\mathcal{I} - \mathcal{F})(\xi)$ *admits no solutions* $\xi \in \partial S$ *for any scalar* $\lambda \in (0, 1/\varepsilon)$.

Then, the operator equation $\mathcal{T}(\xi^*) + \mathcal{F}(\xi^*) = \xi^*$ *possesses at least one solution* $\xi^* \in \overline{S}$.

Definition 2 ([3])**.** *Let* $\mathcal{B}_1$ *and* $\mathcal{B}_2$ *be two Banach spaces. A mapping* $\mathcal{T} : \mathcal{B}_1 \to \mathcal{B}_2$ *is expansive if one can find a constant* $L > 1$ *such that*

$$\|\mathcal{T}(\xi_1) - \mathcal{T}(\xi_2)\|_{\mathcal{B}_2} \geq L\|\xi_1 - \xi_2\|_{\mathcal{B}_1}$$

holds for every pair of points $\xi_1, \xi_2 \in \mathcal{B}_1$.

Finally, we recall the definition of a cone.

Definition 3 ([17])**.** *Let* $C \subset \mathcal{B}$ *be a closed, convex set. Then,* C *is a cone if*

1. $my \in C$ *for any* $m \geq 0$ *and any* $y \in C$,
2. $y, -y \in C$ *implies* $y = 0$.

Denote $C^* = C\backslash\{0\}$. The following fixed point theorem will be instrumental in establishing the existence of a minimum of two nonnegative global classical solutions to the BVP (1). Its proof can be found in [6] and [19].

Theorem 2. *Let* C *be a cone in a Banach space* $\mathcal{B}$*, and let* Ω *be a subset of* C*. Consider three open bounded subsets* $S_1, S_2, S_3 \subset C$ *satisfying*

$$\overline{S}_1 \subset S_2, \quad \overline{S}_2 \subset S_3, \quad \textit{and} \quad 0 \in S_1.$$

Assume the operators $\mathcal{T} : \Omega \to \mathcal{B}$ *and* $\mathcal{F} : \overline{S}_3 \to \mathcal{B}$ *meet the following criteria:*

1. $\mathcal{T}$ *is expansive.*
2. $\mathcal{F}$ *is completely continuous.*
3. $\mathcal{F}(\overline{S}_3) \subset (\mathcal{I} - \mathcal{T})(\Omega)$.
4. *The intersections* $(S_2 \setminus \overline{S}_1) \cap \Omega$ *and* $(S_3 \setminus \overline{S}_2) \cap \Omega$ *are nonempty.*

Furthermore, suppose there exists an element $\xi_0 \in C^*$ *such that*

(a) For all $\lambda > 0$ *and for all* $\xi \in \partial S_1 \cap (\Omega + \lambda\xi_0)$*, we have*

$$\mathcal{F}\xi \neq (\mathcal{I} - \mathcal{T})(\xi - \lambda\xi_0).$$

(b) There is some $\epsilon \geq 0$ *provided that for all* $\lambda \geq 1 + \epsilon$*, for all* $\xi \in \partial S_2$ *with* $\lambda\xi \in \Omega$*, we have*

$$\mathcal{F}\xi \neq (\mathcal{I} - \mathcal{T})(\lambda\xi).$$

(c) For all $\lambda > 0$ and for all $\xi \in \partial S_3 \cap (\Omega + \lambda\xi_0)$, we have

$$\mathcal{F}\xi \neq (\mathcal{I} - \mathcal{T})(\xi - \lambda\xi_0).$$

Then, the operator $\mathcal{T} + \mathcal{F}$ possesses at least two non-zero fixed points $\xi_1, \xi_2 \in C$ such that

$$\xi_1 \in \partial S_2 \cap \Omega \text{ and } \xi_2 \in (\overline{S}_3 \setminus \overline{S}_2) \cap \Omega$$

or

$$\xi_1 \in (S_2 \setminus S_1) \cap \Omega \text{ and } \xi_2 \in (\overline{S}_3 \setminus \overline{S}_2) \cap \Omega.$$

We define the function space X as:

$$X = \left\{ \varphi : J \to \mathbb{R} \mid \varphi \text{ is } rd\text{-continuous on } J_0, \text{ and } \lim_{\xi \to \xi_i^-} \varphi(\xi), \lim_{\xi \to \xi_i^+} \varphi(\xi) \text{ exist with } \varphi(\xi_i^-) = \varphi(\xi_i) \right\} \quad (2)$$

endowed with the norm

$$\|\varphi\| = \sup_{\xi \in J} |\varphi(\xi)|.$$

Definition 4. *By a solution of system* (1)*, we mean a function $\eta(\xi)$ such that η and η^Δ belong to the space X defined above and $\eta(\xi)$ satisfies*

- *the dynamic equation: $\eta^{\Delta^3}(\xi) + f(\xi, \eta(\xi)) = 0$ for all $\xi \in J \setminus \{\xi_1, \ldots, \xi_m\}$;*
- *the jump conditions: $\eta(\xi_j^+) - \eta(\xi_j^-) = I_j(\eta(\xi_j))$, $\quad \eta^\Delta(\xi_j^+) - \eta^\Delta(\xi_j^-) = L_j(\eta(\xi_j))$;*
- *the boundary conditions: $\eta(0) = \eta(\xi_*) = \eta^{\Delta^2}(0) = 0$.*

.

In [9], it is proved that a function $\eta \in X$ is a solution of the BVP (1) if and only if it satisfies the integral equation

$$\eta(\xi) = \int_0^{\xi_*} G(\xi, s) \left(\int_0^s f(\tau, \eta(\tau)) \Delta\tau \right) \Delta s + \sum_{j=1}^{m} W_j(\xi, \eta(\xi_j)), \quad \xi \in [0, \xi_*]_{\mathbb{T}},$$

where

$$G(\xi, s) = \begin{cases} \frac{\xi_* - \xi}{\xi_*} \sigma(s), & 0 \leq s \leq \xi \\ \\ \frac{\xi_* - \sigma(s)}{\xi_*} \xi, & \xi \leq s \leq \xi_*, \end{cases}$$

$$W_j(\xi, \eta(\xi_j)) = \begin{cases} \frac{\xi_* - \xi}{\xi_*} I_j(\eta(\xi_j)) - \frac{\xi_* - \xi}{\xi_*} \xi_j L_j(\eta(\xi_j)), & 0 \leq \xi \leq \xi_j \\ \\ -\frac{\xi}{\xi_*} I_j(\eta(\xi_j)) - \frac{\xi_* - \xi_j}{\xi_*} \xi L_j(\eta(\xi_j)), & \xi_j < \xi \leq \xi_*. \end{cases}$$

Note that

$$0 \leq G(\xi, s) \leq \sigma(\xi_*), \quad \xi, s \in [0, \xi_*]_{\mathbb{T}}.$$

Using $(B2)$, we have

$$|W_j(\xi, \eta(\xi_j))| \leq M\big(c_j + \xi_* d_j\big), \quad j \in \{1, \ldots, m\},$$

where $M := \max_{j\in\{1,2,\ldots m\}} \big\{|\eta(\xi_j)|^{\gamma_j}, |\eta(\xi_j)|^{\beta_j}\big\}$.

Remark 1. *The use of the forward jump operator $\sigma(\xi^*)$ in the upper bound, rather than ξ^*, is necessary to ensure that the inequality remains valid at right-scattered points. By accounting for the time scale's graininess, this approach maintains a uniform bound across all points, including discrete ones.*

The next two sections consist of the main results of this paper.

3 Existence of at Least One Solution

The first main result reads as follows.

Theorem 3. *Assume that $(B1)$ and $(B2)$ hold. Then, the problem (1) has at least one solution in X.*

Proof. For $\eta \in X$, define the operator

$$\mathcal{F}_1\eta(\xi) = \eta(\xi) - \int_0^{\xi_*} G(\xi, s)\bigg(\int_0^s f(\tau, \eta(\tau))\Delta\tau\bigg)\Delta s - \sum_{j=1}^m W_j(\xi, \eta(\xi_j)), \quad \xi \in [0, \xi_*]_{\mathbb{T}}.$$

Note that any $\eta \in X$ that is a solution of the equation

$$\mathcal{F}_1\eta(\xi) = 0, \quad \xi \in [0, \xi_*]_{\mathbb{T}},$$

is a solution to the BVP (1). Define

$$B_1 = B + (\sigma(\xi_*))^3(B + B^{l+1}) + \max_{j\in\{1,2,\ldots m\}}\{B^{\gamma_j}, B^{\beta_j}\}\sum_{j=1}^m (c_j + \xi_* d_j).$$

Lemma 1. *Suppose that $(B1)$ and $(B2)$ hold. Then, for any $\eta \in X$ for which $\|\eta\| \leq B$, one has*

$$|\mathcal{F}_1\eta(\xi)| \leq B_1, \quad \xi \in [0, \xi_*]_{\mathbb{T}}.$$

Proof. Note that

$$\begin{aligned} |f(\xi, \eta(\xi))| &\leq p(\xi) + q(\xi)|\eta(\xi)|^l \\ &\leq B + B^{l+1}, \quad \xi \in [0, \xi_*]_{\mathbb{T}}. \end{aligned}$$

Then

$$|\mathcal{F}_1\eta(\xi)| = \bigg|\eta(\xi) - \int_0^{\xi_*} G(\xi, s)\bigg(\int_0^s f(\tau, \eta(\tau))\Delta\tau\bigg)\Delta s - \sum_{j=1}^m W_j(\xi, \eta(\xi_j))\bigg|$$

$$\begin{aligned} &\leq |\eta(\xi)| + \int_0^{\xi_*} G(\xi,s)\left(\int_0^s |f(\tau,\eta(\tau))|\Delta\tau\right)\Delta s + \sum_{j=1}^{m} |W_j(\xi,\eta(\xi_j))| \\ &\leq B + (\sigma(\xi_*))^3(B + B^{l+1}) + \max_{j\in\{1,2,\ldots m\}}\{B^{\gamma_j}, B^{\beta_j}\}\sum_{j=1}^{m}(c_j + \xi_* d_j) \\ &= B_1. \end{aligned}$$

This completes the proof of Lemma 1. □

In addition, suppose that

(B3) $A\sigma(\xi_*)B_1 \leq B$.

For $\eta \in X$, define the operator

$$\mathcal{F}_2\eta(\xi) = A\int_0^{\xi} \mathcal{F}_1\eta(s)\Delta s, \quad \xi \in [0,\xi_*]_{\mathbb{T}}. \tag{3}$$

Note that if $\eta \in X$ is a solution of the equation

$$\mathcal{F}_2\eta(\xi) = c, \quad \xi \in [0,\xi_*]_{\mathbb{T}}$$

for some constant c, then η is a solution of BVP (1). In fact, if we differentiate both sides of (3) with respect to ξ, we find

$$\mathcal{F}_1\eta(\xi) = 0, \quad \xi \in [0,\xi_*]_{\mathbb{T}}.$$

Lemma 2. *Suppose that* $(B1)$-$(B3)$ *hold. If* $\eta \in X$ *satisfies* $\|\eta\| \leq B$, *then*

$$\|\mathcal{F}_2\eta\| \leq A\sigma(\xi_*)B_1 \leq B.$$

Proof. Applying Lemma 1, we find

$$\begin{aligned} |\mathcal{F}_2\eta(\xi)| &= \left|A\int_0^{\xi} \mathcal{F}_1\eta(s)\Delta s\right| \\ &\leq A\int_0^{\xi} |\mathcal{F}_1\eta(s)|\Delta s \\ &\leq A\sigma(\xi_*)B_1 \\ &\leq B, \quad \xi \in [0,\xi_*]_{\mathbb{T}}. \end{aligned}$$

This completes the proof of Lemma 2. □

Below, with $\widetilde{X}$, we represent the collection of all equicontinuous families in X concerning the norm $\|\cdot\|$. Set Y=$\overline{\widetilde{X}}$ and

$$S = \left\{u \in Y : \|u\| < B \quad \text{and if} \quad \|u\| \geq \frac{B}{2}, \quad \text{then} \quad u(0) > \frac{B}{2}\right\}.$$

Define the operators $\mathcal{T}$ and $\mathcal{F}$ in the following manner

$$\begin{aligned}\mathcal{T}u(\xi) &= \epsilon u(\xi),\\ \mathcal{F}u(\xi) &= u(\xi) - \epsilon u(\xi) - \epsilon\mathcal{F}_2 u(\xi)\end{aligned}$$

for $u \in \overline{S},\ \epsilon > 0$ and $\xi \in [0, \xi_*]_{\mathbb{T}}$. Fix $u \in \overline{S}$. Then, we get

$$\begin{aligned}\|(\mathcal{I} - \mathcal{F})u\| &= \|\epsilon u + \epsilon\mathcal{F}_2 u\|\\ &\leq \epsilon\|u\| + \epsilon\|\mathcal{F}_2 u\|\\ &\leq \epsilon B_1 + \epsilon A B_1.\end{aligned}$$

Therefore, $\mathcal{F} : \overline{S} \to X$ is a continuous operator and $(\mathcal{I} - \mathcal{F})(\overline{S})$ lies within a compact subset of Y. Below, assume that $u \in \partial S$ satisfies the equation

$$u = \lambda(\mathcal{I} - \mathcal{F})u,$$

or u is a solution to the following equation

$$u = \lambda\epsilon\,(u + \mathcal{F}_2 u)$$

for some $\lambda \in \left(0, \frac{1}{\epsilon}\right)$. Observe that $\mathcal{F}_2 u(0) = 0$. Since $\|u\| = B > \frac{B}{2}$, we conclude that $u(0) > \frac{B}{2}$, and

$$u(0) = \lambda\epsilon u(0).$$

Therefore $\lambda\epsilon = 1$. This is a contradiction. Consequently

$$\left\{u \in \partial S : u = \lambda_1(\mathcal{I} - \mathcal{F})u\right\} = \emptyset$$

for any $\lambda_1 \in \left(0, \frac{1}{\epsilon}\right)$. Now, employing Theorem 1, we conclude that the operator $\mathcal{T} + \mathcal{F}$ has a fixed point $u^* \in Y$. Thus,

$$\begin{aligned}u^*(\xi) &= \mathcal{T}u^*(\xi) + \mathcal{F}u^*(\xi)\\ &= \epsilon u^*(\xi) - u^*(\xi) + \epsilon u^*(\xi) - \epsilon\mathcal{F}_2 u^*(\xi), \quad \xi \in [0, \xi_*]_{\mathbb{T}}.\end{aligned}$$

Hence,

$$0 = \mathcal{F}_2 u^*(\xi), \quad \xi \in [0, \xi_*]_{\mathbb{T}}.$$

From this, we find that u^* is a solution to problem (1) and the proof of Theorem 3 is completed. □

4 Existence of at Least Two Solutions

In this section, applying the new techniques presented in Section 2, we prove the following result.

Theorem 4. *Assume that* $(B1)$ *and* $(B2)$ *hold. We may therefore conclude that there exist at least two distinct nonnegative solutions to* (1) *in the function space* X.

Proof. Let X denote the Banach space defined by (2), and suppose r_1, ℓ, and R_1 are positive constants subject to the constraints $\ell < R_1$, $R_1 \geq \frac{1}{2}(r_1 + \ell)$ and

$$r_1 < \ell < R_1 \leq B,\ AB_1\sigma(\xi_*) < \frac{\ell}{5}.$$

Denote

$$\widetilde{C} = \{u \in X : u \geq 0 \quad \text{on} \quad [0, \xi_*]_{\mathbb{T}}\}.$$

Let C be the set of all equicontinuous families in $\widetilde{C}$. For $v \in X$ and $\xi \in [0, \xi_*]_{\mathbb{T}}$, define the operators $\mathcal{T}_1$ and $\mathcal{F}_3$ in the following manner

$$\begin{aligned} \mathcal{T}_1 v(\xi) &= (1 + m\epsilon)v(\xi) - \epsilon\frac{\ell}{10}, \\ \mathcal{F}_3 v(\xi) &= -\epsilon\mathcal{F}_2 v(\xi) - m\epsilon v(\xi) - \epsilon\frac{\ell}{10}. \end{aligned}$$

Observe that any fixed point $v \in X$ of the operator $\mathcal{T}_1 + \mathcal{F}_3$ is a solution to the BVP (1). Define

$$\begin{aligned} S_1 &= C_{r_1} = \{v \in C : \|v\| < r_1\}, \\ S_2 &= C_\ell = \{v \in C : \|v\| < \ell\}, \\ S_3 &= C_{R_1} = \{v \in C : \|v\| < R_1\}, \\ \Omega &= C. \end{aligned}$$

1. Fix $v_1, v_2 \in \Omega$. Then, we obtain

$$\|\mathcal{T}_1 v_1 - \mathcal{T}_1 v_2\| = (1 + m\epsilon)\|v_1 - v_2\|.$$

Therefore, the operator $\mathcal{T}_1 : \Omega \to X$ is an expansive operator with a constant $L = 1 + m\epsilon > 1$.

2. Take $v \in \overline{C_{R_1}}$ arbitrarily. Then, we arrive at

$$\begin{aligned} \|\mathcal{F}_3 v\| &\leq \epsilon\|\mathcal{F}_2 v\| + m\epsilon\|v\| + \epsilon\frac{\ell}{10} \\ &\leq \epsilon\left(A\sigma(\xi_*)B_1 + mR_1 + \frac{\ell}{10}\right). \end{aligned}$$

Consequently, $\mathcal{F}_3(\overline{C_{R_1}})$ is uniformly bounded. Now, using that $\mathcal{F}_3 : \overline{C_{R_1}} \to X$ is a continuous operator, we obtain that $\mathcal{F}_3(\overline{C_{R_1}})$ is equicontinuous. Therefore $\mathcal{F}_3 : \overline{C_{R_1}} \to X$ is completely continuous.

3. Fix $v_1 \in \overline{C_{R_1}}$ and denote

$$v_2 = v_1 + \frac{1}{m}\mathcal{F}_2 v_1 + \frac{\ell}{5m}.$$

Observe that $\mathcal{F}_2 v_1 + \frac{\ell}{5} \geq 0$ on $[0, \xi_*]_{\mathbb{T}}$. Then, $v_2 \geq 0$ on $[0, \xi_*]_{\mathbb{T}}$ and $v_2 \in \Omega$, and

$$-\epsilon m v_2 = -\epsilon m v_1 - \epsilon \mathcal{F}_2 v_1 - \epsilon \frac{\ell}{10} - \epsilon \frac{\ell}{10}$$

or

$$\begin{aligned} (\mathcal{I} - \mathcal{T}_1) v_2 &= -\epsilon m v_2 + \epsilon \frac{\ell}{10} \\ &= \mathcal{F}_3 v_1. \end{aligned}$$

Thus, we obtain $\mathcal{F}_3(\overline{C_{R_1}}) \subset (\mathcal{I} - \mathcal{T}_1)(\Omega)$.

4. Suppose that for any $v_0 \in C^* = C \backslash \{0\}$, there are $\lambda > 0$ and $v \in \partial C_{r_1} \cap (\Omega + \lambda v_0)$ or $v \in \partial C_{R_1} \cap (\Omega + \lambda v_0)$ such that

$$\mathcal{F}_3 v = (\mathcal{I} - \mathcal{T}_1)(v - \lambda v_0).$$

Hence, we find the following

$$-\epsilon \mathcal{F}_2 v - m\epsilon v - \epsilon \frac{\ell}{10} = -m\epsilon(v - \lambda v_0) + \epsilon \frac{\ell}{10}$$

or

$$-\mathcal{F}_2 v = \lambda m v_0 + \frac{\ell}{5}.$$

Therefore,

$$\|\mathcal{F}_2 v\| = \left\| \lambda m v_0 + \frac{\ell}{5} \right\| > \frac{\ell}{5}.$$

This is a contradiction.

5. Take $\epsilon_1 = \frac{2}{5m}$ and assume that there are $v_1 \in \partial C_\ell$ and $\lambda_1 \geq 1 + \epsilon_1$ for which we have the following

$$\mathcal{F}_3 v_1 = (\mathcal{I} - \mathcal{T}_1)(\lambda_1 v_1).$$

In addition, we find

$$-\epsilon \mathcal{F}_2 v_1 - m\epsilon v_1 - \epsilon \frac{\ell}{10} = -\lambda_1 m \epsilon v_1 + \epsilon \frac{\ell}{10}$$

or

$$\mathcal{F}_2 v_1 + \frac{\ell}{5} = (\lambda_1 - 1) m v_1.$$

Hence,

$$2\frac{\ell}{5} > \left\| \mathcal{F}_2 v_1 + \frac{\ell}{5} \right\| = (\lambda_1 - 1) m \|v_1\| = (\lambda_1 - 1) m \ell$$

and

$$\frac{2}{5m} + 1 > \lambda_1.$$

This is a contradiction.

Consequently, all conditions of Theorem 3 hold and we conclude that the problem (1) has at least two solutions u_1 and u_2. For these solutions, we have

$$\|u_1\| = \ell < \|u_2\| \leq R_1$$

or

$$r_1 \leq \|u_1\| < \ell < \|u_2\| \leq R_1.$$

□

5 An Example

Let $\mathbb{T} = \bigcup_{j=0}^{3} [2j, 2j+1]$, where each interval $[2j, 2j+1]$, $j \in \{0, \dots 3\}$, is real-valued. Let also $\xi_* = 7$, $B = 1$, $m = 3$, $p(\xi) = \frac{1}{1000}(\xi^2 + \xi + 1)$, $q(\xi) = \frac{1}{10}$, $\xi \in [0, 7]$, $c_1 = 1$, $c_2 = c_3 = 2$, $d_1 = d_2 = 1$, $d_3 = 2$, $\gamma_1 = 2$, $\gamma_2 = 1$, $\gamma_3 = 0$, $\beta_1 = 1$, $\beta_2 = 0$, $\beta_3 = 1/2$, and

$$r_1 = \frac{1}{10}, \quad \ell = \frac{1}{5}, \quad R_1 = \frac{1}{2}, \quad A = \frac{1}{10B_1}, \quad l = 4,$$

and

$$\xi_0 = 0, \quad \xi_1 = 2, \quad \xi_2 = 4, \quad \xi_3 = 6,$$

and

$$f(\xi, \eta) = \frac{1}{1000}(\xi^2 + \xi + 1) + \frac{1}{10}\eta^4, \quad \xi \in [0, 7], \quad \eta \in \mathbb{R}.$$

Then

$$p(\xi) \leq 1, \quad q(\xi) \leq 1, \quad AB_1\sigma(\xi_*) = \frac{7}{10} < 1, \quad r_1 < \ell < R_1 \leq B.$$

Thus, $(B1)$, $(B2)$, and $(B3)$ hold. Consider the following BVP

$$\begin{cases} \eta^{\Delta^3}(\xi) + \frac{1}{1000}(\xi^2 + \xi + 1) + \frac{1}{10}(\eta(\xi))^4 &= 0, \quad \xi \in J \backslash \{\xi_1, \xi_2, \xi_3\}, \\ \eta(\xi_1^+) - \eta(\xi_1) &= \eta^2, \\ \eta(\xi_2^+) - \eta(\xi_2) &= 2\sin\eta, \\ \eta(\xi_3^+) - \eta(\xi_3) &= 2, \\ \eta^{\Delta}(\xi_1^+) - \eta^{\Delta}(\xi_1) &= \frac{\eta}{1+\eta^2}, \\ \eta^{\Delta}(\xi_2^+) - \eta^{\Delta}(\xi_2) &= 1, \\ \eta^{\Delta}(\xi_3^+) - \eta^{\Delta}(\xi_3) &= 2\sqrt{\eta}, \\ \eta(0) = \eta(7) = \eta^{\Delta^2}(0) &= 0. \end{cases} \tag{4}$$

Thus, for BVP (4), Theorem 3 and Theorem 4 hold.

Figure 1 illustrate the plots of two distinct solutions over the specified time scale. It is observed that the solutions remain non-negative and satisfy the given boundary conditions.

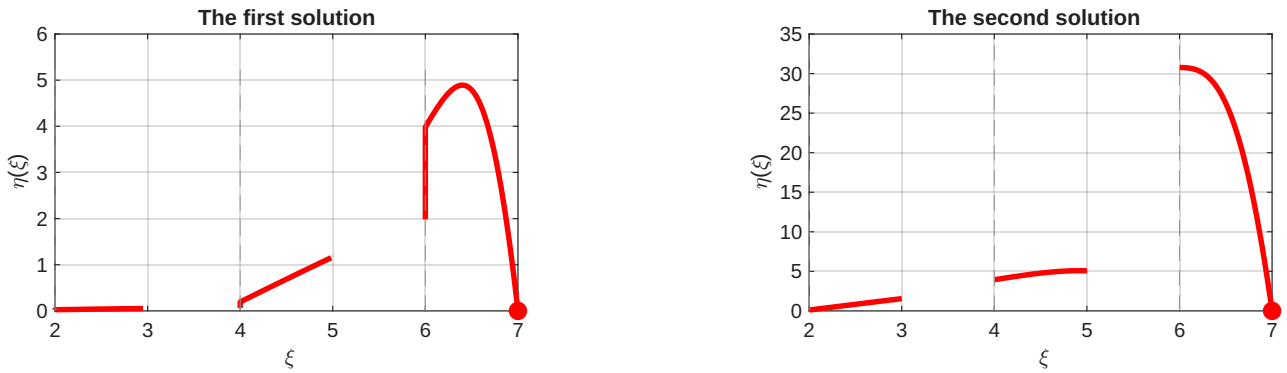


Figure 1: Illustrations for solutions of Equation (4).

6 Conclusion

This study has successfully addressed a significant gap in the qualitative theory of dynamic equations on time scales by establishing novel existence criteria for solutions to a class of third-order impulsive nonlinear boundary value problems. The primary challenge in analyzing such problems lies in the interplay between the higher-order dynamics, the impulsive effects causing discontinuous state jumps, and the complex structure of general time domains.

Our approach was grounded in advanced fixed point theory, specifically leveraging the properties of expansive mappings and the decomposition of operators. This methodology provided a powerful and flexible framework to overcome the limitations of traditional techniques. The main results of this paper, Theorems 3 and 4, conclusively demonstrate the existence of at least one and, under further conditions, at least two distinct solutions to the BVP (1). A paramount advantage of our approach is that it circumvents the need for any boundedness or sublinear growth conditions on the nonlinear term f or the impulsive functions I_j and L_j, a restrictive requirement that is commonplace in much of the existing literature. This significantly broadens the class of problems to which our theory can be applied, as evidenced by the illustrative example provided in Section 5.

In summary, this work makes several key contributions:

- It extends the current literature on impulsive dynamic equations by focusing on the underdeveloped area of third-order problems.
- It introduces a novel application of recent fixed point theorems involving operator addition and expansive maps to this class of BVPs.
- It derives existence results under weaker and more general assumptions, particularly by eliminating boundedness constraints.
- It provides a concrete example that validates the applicability of our theoretical findings.

As a result, the findings presented herein provide a new perspective for analyzing complex systems with mixed continuous-discrete evolution and sudden perturbations. Natural extensions of this work include investigating the existence of positive solutions, exploring the case of infinite time intervals or infinite impulsive points, and analyzing the stability properties of the solutions. Furthermore, applying this methodology to more specific applied models in physics, biology, or engineering represents a fertile ground for future study.

Author Contribution Statements

Both authors contributed equally to this work.

Declaration of Competing Interests

They have no conflict of interest.

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No data were used or generated during the conduct of this study.

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Ethical approval was not required for this study as it does not involve human subjects or animals.

Declaration of Generative Artificial Intelligence Use

During the preparation of this work, the authors used generative AI technologies to improve language and correct typography. After using these technologies, the authors reviewed and edited the content as needed and take full responsibility for the content of the publication.

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References

[1] Agarwal, R. P. and Bohner, M. (1999). Basic calculus on time scales and some of its applications. *Results in Mathematics*, 35(1):3–22.

[2] Agarwal, R. P., Hazarika, B., and Tikare, S. (2024). *Dynamic Equations on Time Scales and Applications*. Chapman and Hall/CRC, 1 edition.

[3] Avramescu, C. and Vladimirescu, C. (2003). Some remarks on Krasnoselskii's fixed point theorem. *Fixed Point Theory*, 4:3–13.

[4] Bohner, M. and Peterson, A. (2001). *Dynamic Equations on Time Scales: An Introduction with Applications*. Springer Science & Business Media.

[5] Bohner, M. and Peterson, A. (2002). *Advances in Dynamic Equations on Time Scales*. Springer Science & Business Media.

[6] Djebali, S. and Mebarki, K. (2019). Fixed point index for expansive perturbation of k-set contraction mappings. *Topological Methods in Nonlinear Analysis*, 54(2):613–640.

[7] Dugundji, J. and Granas, A. (1982). *Fixed Point Theory*, volume 1 of *Monografie Matematyczne*. PNW, Warsaw.

[8] Georgiev, S., Kheloufi, A., and Mebarki, K. (2024). Existence of classical solutions for a class of impulsive hamilton-jacobi equations. *Palestine Journal of Mathematics*, 13(4):1084–1097.

[9] Georgiev, S. and Zennir, K. (2021). *Boundary Value Problems on Time Scales*, volume II. CRC Press.

[10] Georgiev, S. G. (2024). Existence of solutions for a class of wave dynamic equations on time scales. *Discrete and Continuous Dynamical Systems – Series S*, 17(10):2967–2981.

[11] Georgiev, S. G., Akgöl, S. D., and Kuş, M. E. (2022). Existence of solutions for third order multi point impulsive boundary value problems on time scales. *Miskolc Mathematical Notes*, 23(2):677–690.

[12] Georgiev, S. G., Tikare, S., and Kumar, V. (2023). Existence of solutions for two-point integral boundary value problems with impulses. *Qualitative Theory of Dynamical Systems*, 22:97.

[13] Guan, W. and Wang, D. B. (2014). Positive solutions for third-order p-Laplacian functional dynamic equations on time scales. *Abstract and Applied Analysis*, page 676052.

[14] Karaca, I. Y. and Fen, F. T. (2016a). Existence of positive solutions for nonlinear third-order m-point impulsive boundary-value problems on time scales. *Ukrainian Mathematical Journal*, 68:458–474.

[15] Karaca, I. Y. and Fen, F. T. (2016b). On positive solutions of nonlinear third-order impulsive boundary value problems on time scales. *Mediterranean Journal of Mathematics*, 13:4447–4461.

[16] Khemmar, K., Mebarki, K., and Georgiev, S. (2024). Existence of solutions for a class of boundary value problems for weighted $p(x)$-Laplacian impulsive systems. *Filomat*, 38(21):7563–7577.

[17] Krasnosel'skii, M. A. (1964). *Positive Solutions of Operator Equations*. Wolters-Noordhoff.

[18] Li, Y. and Li, Y. (2011). Existence of solutions of boundary value problems for a nonlinear third-order impulsive dynamic system on time scales. *Differential Equations and Applications*, 3(3):309–322.

[19] Mouhous, M., Georgiev, S. G., and Mebarki, K. (2022). Existence of solutions for a class of first order boundary value problems. *Archivum Mathematicum*, 58:141–158.

[20] Qiu, Y. C. (2014). Nonoscillatory solutions to third-order neutral dynamic equations on time scales. *Advances in Difference Equations*, 2014(1):309.

[21] Qiu, Y. C., Jadlovská, I., Chiu, K. S., and Li, T. (2020). Existence of nonoscillatory solutions tending to zero of third-order neutral dynamic equations on time scales. *Advances in Difference Equations*, 2020:231.

[22] Scindia, P., Tikare, S., and El-Deeb, A. A. (2023). Ulam stability of first-order nonlinear impulsive

dynamic equations. *Boundary Value Problems*, 2023(1):86.

[23] Sun, J. P. (2006). Existence of solution and positive solution of bvp for nonlinear third-order dynamic equation. *Nonlinear Analysis*, 64(3):629–636.

[24] Tikare, S. and Tisdell, C. C. (2020). Nonlinear dynamic equations on time scales with impulses and nonlocal conditions. *Journal of Classical Analysis*, 16(2):125–140.

[25] Xu, H. (2010). Existence and uniqueness of positive solutions for a third-order three-point problem on time scales. *Electronic Journal of Qualitative Theory of Differential Equations*, 2010(56):1–12.

[26] Şenel, M. T. (2013). Behavior of solutions of a third-order dynamic equation on time scales. *Journal of Inequalities and Applications*, 2013(1):47.